Max-Min Fairness Linear Transceiver Design for a Multi-User MIMO Interference Channel

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***Abstract*—Consider the max-min fairness linear transceiver design problem for a multi-user MIMO interference channel. Assuming perfect channel knowledge, this problem can be formulated as the maximization of the minimum SINR utility, subject to individual power constraints at the transmitters. In this paper, it is shown that when the number of antennas is at least three at each transmitter (receiver) and is at least two at each receiver (transmitter), the problem of checking the feasibility of a given set of target SINR levels is strongly NP-hard. A cyclic coordinate ascent algorithm is proposed for this problem. Monotonicity and global convergence (to a KKT solution) are established for the proposed algorithm. Numerical simulations show the proposed algorithm is efficient by using the channel matched beamformer as the benchmark.**

I. Introduction

Consider a multi-user multi-input multi-output (MIMO) interference channel, where K independent data streams are transmitted to K receivers. Both the transmitters and the receivers are equipped with multiple antennas, and are assumed to share the same frequency band/time slot. To mitigate multi-user interference and maximize throughput, we consider a joint power control and transmit-receive beamformer design problem subject to individual power budget constraints. To ensure user fairness, we choose the minimum signal-to-interference-and-noise ratio (SINR) as the system utility function. This max-min fairness utility has been considered in previous studies of the transceiver design and power control problem [1], [2], where the authors proposed to approximate the optimum by minimizing the sum of equally weighted inverse signal-to-interference ratios (SIR). In the case of single receive antenna per receiver, the authors of [3] further extended this approach by choosing suitable weight factors with which the weighted sum-SIR maximization can achieve optimal max-min fairness. Also, polynomial time algorithms capable of achieving global optimality have been provided in [4] (also see [5], [6]) for the power control and/or transmit beamforming design problems, again for the single receive antenna case.

In this paper, we consider the case that the number of antennas is at least three at each transmitter (receiver) and is at least two at each receiver (transmitter). In this case, we show that even the problem of checking the feasibility of a given set of target SINRs is strongly NP-hard. This result is in sharp contrast to the polynomial time solvability of the max-min SINR beamforming problem in the single receive antenna case [4]–[6]. Motivated by this NP-hardness result, we focus our effort to find locally optimal beamformers for the max-min SINR problem. Due to the separability of the beamforming vectors, we propose a cyclic coordinate ascent algorithm, different from the uplink-downlink duality in [7]–[9], to solve the max-min SINR problem in the multi-antenna case, where in each step a simple convex optimization problem is solved. Such a cyclic optimization procedure can be implemented in a distributed fashion.

We adopt the following notations in this paper. Lowercase boldface and uppercase boldface are used for vectors and matrices. For a given matrix $\mathbf{H}$, $\mathbf{H}^{\mathrm{T}}$ and $\mathbf{H}^{\dagger}$ are the transpose and conjugate transpose of $\mathbf{H}$. The notation $\mathbf{A} \succeq \mathbf{B}$ ($\mathbf{A} \succ \mathbf{B}$) means that $\mathbf{A}-\mathbf{B}$ is a positive semi-definite (definite) Hermitian matrix and $\mathbf{I}$ represents the identity matrix of an appropriate size. We use $\mathbf{e}_n^k$ to denote a n-dimensional column vector with its k-th element being one and other elements zero. Finally, we use $\mathcal{K} \triangleq \{1, 2, ..., K\}$ to denote the set of users.

II. Problem Formulation

Consider a K-user MIMO interference channel where the k-th transmitter and receiver are equipped with N_k and M_k antennas respectively. For the single-carrier channel, the received signal at receiver k is

$$\mathbf{y}_k = \mathbf{H}_{kk}\mathbf{v}_k s_k + \sum_{j\neq k}\mathbf{H}_{kj}\mathbf{v}_j s_j + \mathbf{z}_k, \tag{1}$$

where $\mathbf{H}_{kj} \in \mathbb{C}^{M_k\times N_j}$ is the channel matrix between transmitter j and receiver k, $\mathbf{v}_k$ is the beamformer used by transmitter k, s_k is the symbol that transmitter k wishes to send to receiver k, and $\mathbf{z}_k$ is the additive white Gaussian noise (AWGN) with distribution $\mathcal{CN}(\mathbf{0}, \sigma_k^2\mathbf{I})$. Let $\mathbf{u}_k$ be the receive beamformer for receiver k and a linear reception strategy is assumed. Treating interference as noise, the SINR of user k can be written as

$$\mathrm{SINR}_k = \frac{|\mathbf{u}_k^{\dagger}\mathbf{H}_{kk}\mathbf{v}_k|^2}{\sigma_k^2\|\mathbf{u}_k\|^2 + \sum_{j\neq k}|\mathbf{u}_k^{\dagger}\mathbf{H}_{kj}\mathbf{v}_j|^2}. \tag{2}$$

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The linear transceiver design problem is further formulated as

$$\begin{array}{ll} \max & U(\mathrm{SINR}_1, \mathrm{SINR}_2, ..., \mathrm{SINR}_K) \\ \text{s.t.} & \|\mathbf{u}_k\| = 1,\ \|\mathbf{v}_k\|^2 \le P_k,\ k \in \mathcal{K}, \end{array} \tag{3}$$

where P_k denotes the power budget of transmitter k and $U(\cdot)$ the system utility. A special case of (3) is the popular sum-rate maximization problem [5], [6], [9], where

$$U(\mathrm{SINR}_1, \mathrm{SINR}_2, ..., \mathrm{SINR}_K) = \sum_{k\in\mathcal{K}} \log\left(1 + \mathrm{SINR}_k\right).$$

While sum-rate maximization achieves high network throughput, it typically causes unfairness among users. A more fair strategy is to use the max-min fairness criterion as follows:

$$\begin{array}{ll} \max & G(\mathbf{u}, \mathbf{v}) \triangleq \min\limits_{k\in\mathcal{K}} \mathrm{SINR}_k \\ \text{s.t.} & \|\mathbf{u}_k\| = 1,\ \|\mathbf{v}_k\|^2 \le P_k,\ k \in \mathcal{K}, \end{array} \tag{4}$$

which is equivalent to

$$\begin{array}{ll} \max & \mathrm{SINR} \\ \text{s.t.} & \mathrm{SINR} \le \mathrm{SINR}_k, \|\mathbf{u}_k\| = 1, \|\mathbf{v}_k\|^2 \le P_k,\ k \in \mathcal{K}. \end{array} \tag{5}$$

III. Complexity Analysis

In this section, we investigate the complexity status of the optimization problem (5) and the problem of checking the feasibility of a given set of SINR levels.

When each receiver has a single antenna, the max-min SINR fairness problem (5) can be solved in polynomial time via the bisection technique ([4] and in [6, Theorem 3.3]), where each step solves a second order cone programming [10]; when each transmitter is equipped with a single antenna, problem (5) can also be solved in polynomial time via solving a series of semi-definite programming (SDP) [11]. In particular, for the single transmit antenna case, channel matrices $\mathbf{H}_{kj}$ in (2) reduce to column channel vectors $\mathbf{h}_{kj}$, and we replace $\|\mathbf{v}_k\|^2$ in (5) with x_k. Noting that the optimal receive beamformer is the linear minimum mean square error (LMMSE) receive beamformer and letting $\xi = 1/\mathrm{SINR}$, we can eliminate the receive beamformers $\mathbf{u}_k$ from (5) and transform it into

$$\begin{array}{ll} \min & \xi \\ \text{s.t.} & x_k(1+\xi)\mathbf{h}_{kk}^\dagger \left(\sigma_k^2 \mathbf{I} + \sum\limits_{j=1}^K \mathbf{h}_{kj}\mathbf{h}_{kj}^\dagger x_j\right)^{-1} \mathbf{h}_{kk} \ge 1, \\ & 0 \le x_k \le P_k,\ k\in\mathcal{K}. \end{array} \tag{6}$$

Consider the following optimization problem

$$\begin{array}{ll} \max & \xi \\ \text{s.t.} & x_k(1+\xi)\mathbf{h}_{kk}^\dagger \left(\sigma_k^2 \mathbf{I} + \sum\limits_{j=1}^K \mathbf{h}_{kj}\mathbf{h}_{kj}^\dagger x_j\right)^{-1} \mathbf{h}_{kk} \le 1, \\ & 0 \le x_k \le P_k,\ k\in\mathcal{K}. \end{array} \tag{7}$$

Using a contra-positive argument similar to that of [4], [7], we can show that (6) and (7) have the same optimal solution which is attained when all the SINR constraints are met with equality. This implies that (6) and (7) are equivalent to each other. Moreover, the SINR constraints in (7) are equivalent to

$$\sigma_k^2 \mathbf{I} + \sum_{j=1}^K \mathbf{h}_{kj}\mathbf{h}_{kj}^\dagger x_j \succeq x_k(1+\xi)\mathbf{h}_{kk}\mathbf{h}_{kk}^\dagger,\ k\in\mathcal{K}.$$

Given a ξ, we only need to solve an SDP to check its feasibility.

Theorem 3.1: When each transmitter is equipped with a single antenna ($N_k = 1, k \in \mathcal{K}$), the linear transceiver design problem (5) is polynomial time solvable; when each receiver is equipped with a single antenna ($M_k = 1, k \in \mathcal{K}$), problem (5) can also be solved in polynomial time.

We now examine the situation where all transmitters and receivers are equipped with multiple antennas.

Theorem 3.2: Given target minimum SINR $= \zeta$ in (5), the problem of checking the feasibility of ζ is strongly NP-hard when each transmitter (receiver) is equipped with at least three (two) antennas and each receiver (transmitter) is equipped with at least two (three) antennas.

Proof: Without loss of generality, we consider the case $M_k = 3,\ N_k = 2,\ k\in\mathcal{K}$. The proof is based on a polynomial time transformation from the 3SAT problem, which is known to be NP-complete [12]. The 3SAT problem is described as follows: given m disjunctive clauses[1] defined on n Boolean variables such that each clause contains exactly three literals, the question is to check whether there exists a truth assignment for these Boolean variables such that all clauses are satisfied.

Given any instance of the 3SAT problem consisting of m disjunctive clauses $c_1, c_2, ..., c_m$ defined on n Boolean variables $x_1, x_2, ..., x_n$, we construct below a MIMO interference channel with $\mathcal{K} = \{1, 2, ..., n+m\}$ and the channel matrices $\mathbf{H}_{kj} \in \mathbb{R}^{3\times 2}, \forall\ k, j \in \mathcal{K}$. We first define

$$\mathbf{H}_A = M\mathbf{e}_3^1\left(\mathbf{e}_2^1\right)^{\mathrm{T}}, \mathbf{H}_B = M\mathbf{e}_3^1\left(\mathbf{e}_2^2\right)^{\mathrm{T}}, \mathbf{H}_C = M\mathbf{e}_3^2\left(\mathbf{e}_2^1\right)^{\mathrm{T}},$$
$$\mathbf{H}_D = M\mathbf{e}_3^2\left(\mathbf{e}_2^2\right)^{\mathrm{T}}, \mathbf{H}_E = M\mathbf{e}_3^3\left(\mathbf{e}_2^1\right)^{\mathrm{T}}, \mathbf{H}_F = M\mathbf{e}_3^3\left(\mathbf{e}_2^2\right)^{\mathrm{T}},$$

where $M = 6\sqrt{2}$. All of direct-link matrices are set to be

$$\mathbf{H}_{kk} = \mathbf{E} \triangleq \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{pmatrix},\ k \in\ \mathcal{K}.$$

As for the crosstalk channels, the corresponding channel matrices are: for user $k,\ k = 1, 2, ..., n$, let $\mathbf{H}_{kj} = \mathbf{0},\ \forall\ j \in \mathcal{K}\backslash k$; and for user $k,\ k = n+1, n+2, ..., n+m$, $\mathbf{H}_{kj} = \mathbf{0}$ except

$$\mathbf{H}_{kj} = \begin{cases} \mathbf{H}_B, & \text{if } \alpha_{\pi(k)} = x_j \text{ for some } j; \\ \mathbf{H}_A, & \text{if } \alpha_{\pi(k)} = \bar{x}_j \text{ for some } j, \end{cases}$$

$$\mathbf{H}_{kj} = \begin{cases} \mathbf{H}_D, & \text{if } \beta_{\rho(k)} = x_j \text{ for some } j; \\ \mathbf{H}_C, & \text{if } \beta_{\rho(k)} = \bar{x}_j \text{ for some } j, \end{cases}$$

$$\mathbf{H}_{kj} = \begin{cases} \mathbf{H}_F, & \text{if } \gamma_{\tau(k)} = x_j \text{ for some } j; \\ \mathbf{H}_E, & \text{if } \gamma_{\tau(k)} = \bar{x}_j \text{ for some } j, \end{cases}$$

[1] For a given set of Boolean variables, a literal is defined as either a Boolean variable or its negation, while a disjunctive clause refers to a logical expression consisting of the logical "OR" of literals.

where $c_{k-n} = \alpha_{\pi(k)} \vee \beta_{\rho(k)} \vee \gamma_{\tau(k)}$, α, β, and γ are taken from $\{x, \bar{x}\}$ and π, ρ, and τ are mappings from $\{n+1, n+2, ..., n+m\}$ to $\{1, 2, ..., n\}$. Let $\sigma_k = 1, P_k = 1 (k \in \mathcal{K})$ and $\zeta = 1$.The constructed feasibility instance of problem (5) is

$$\begin{cases} \text{SINR}_k \geq 1,\ k \in \mathcal{K}; \\ \|\mathbf{u}_k\| = 1,\ k \in \mathcal{K}; \\ \|\mathbf{v}_k\|^2 \leq 1,\ k \in \mathcal{K}, \end{cases} \tag{8}$$

where

$$\text{SINR}_k = \begin{cases} |\mathbf{u}_k^\dagger \mathbf{E} \mathbf{v}_k|^2, \text{ if } k = 1, 2, ..., n; \\ \dfrac{|\mathbf{u}_k^\dagger \mathbf{E} \mathbf{v}_k|^2}{1 + \sum_{f \in \{\pi, \rho, \tau\}} |\mathbf{u}_k^\dagger \mathbf{H}_{kf(k)} \mathbf{v}_{f(k)}|^2}, \text{ otherwise.} \end{cases}$$

Let $\mathbf{v}_k = (x_k, y_k)^{\text{T}}$. Next, we show *the 3SAT problem is satisfied if and only if there exists a feasible solution to* (8).

If there exists a truth assignment such that all the clauses in the 3SAT problem are satisfied, we prove that there exist $\mathbf{u}_k, \mathbf{v}_k\ (k \in \mathcal{K})$ satisfying all the conditions in (8). Let

$$\begin{cases} y_k = 1 - x_k, & \text{if } k = 1, 2, ..., n; \\ x_k = 1,\ y_k = 0, & \text{if } k = n+1, n+2, ..., n+m. \end{cases} \tag{9}$$

Hence we get a group of feasible $\mathbf{v}_k$ (either $\mathbf{e}_2^1$ or $\mathbf{e}_2^2$) for (8). We claim these beamforming vectors make the dimension of the interference subspace at each receiver at most two. As an illustrative example, suppose that $c_1 = x_1 \vee \bar{x}_2 \vee x_3$ is satisfied. Therefore, at least *one* of the following facts $x_1 = 1$, $x_2 = 0$, or $x_3 = 1$ hold true. Consider receiver $n+1$. The interference vectors at receiver $n+1$ are

$$\begin{aligned} \mathcal{I}_{n+1,1} &= \mathbf{H}_{n+1,1}\mathbf{v}_1 = \mathbf{H}_B \mathbf{v}_1 = M y_1 \mathbf{e}_3^1, \\ \mathcal{I}_{n+1,2} &= \mathbf{H}_{n+1,2}\mathbf{v}_2 = \mathbf{H}_C \mathbf{v}_2 = M x_2 \mathbf{e}_3^2, \\ \mathcal{I}_{n+1,3} &= \mathbf{H}_{n+1,3}\mathbf{v}_3 = \mathbf{H}_F \mathbf{v}_3 = M y_3 \mathbf{e}_3^3. \end{aligned} \tag{10}$$

Thus, the dimension of the interference subspace at receiver $n+1$ is at most two. The same argument can be used for the other users. Moreover, $\mathbf{u}_k\ (k = n+1, n+2, ..., n+m)$ can be chosen from $\{\mathbf{e}_3^1, \mathbf{e}_3^2, \mathbf{e}_3^3\}$ such that the leakage interference at receiver k is zero, i.e., for $k = n+1, n+2, ..., n+m$,

$$\mathbf{u}_k^\dagger \mathbf{H}_{kj} \mathbf{v}_j = 0,\ \forall\ j \neq k. \tag{11}$$

Set $\mathbf{u}_k = \mathbf{e}_3^1\ (k = 1, 2, ..., n)$ and check SINRs of all users:

- for user k, $k = 1, 2, ..., n$,

$$\text{SINR}_k = |\mathbf{u}_k^\dagger \mathbf{H}_{kk} \mathbf{v}_k|^2 = |\left(\mathbf{e}_3^1\right)^\dagger \mathbf{E} \mathbf{v}_k|^2 = 1$$

  due to $\mathbf{v}_k \in \{\mathbf{e}_2^1, \mathbf{e}_2^2\}$;
- for user k, $k = n+1, n+2, ..., n+m$,

$$\begin{aligned} \text{SINR}_k &= \frac{|\mathbf{u}_k^\dagger \mathbf{H}_{k,k} \mathbf{v}_k|^2}{1 + \sum_{f \in \{\pi, \rho, \tau\}} |\mathbf{u}_k^\dagger \mathbf{H}_{kf(k)} \mathbf{v}_{f(k)}|^2} \\ &= |\mathbf{u}_k^\dagger \mathbf{H}_{k,k} \mathbf{v}_k|^2 \quad \text{(from (11))} \\ &= |\mathbf{u}_k^\dagger \mathbf{E} \mathbf{e}_2^1|^2 \quad \text{(from (9))} \\ &= 1. \quad (\mathbf{u}_k \in \{\mathbf{e}_3^1, \mathbf{e}_3^2, \mathbf{e}_3^3\}) \end{aligned}$$

As a result, if there exists a truth assignment satisfying all the clauses in the given 3SAT problem, then we can construct a beamforming vector $\mathbf{u}_k, \mathbf{v}_k\ (k \in \mathcal{K})$ satisfying (8).

For the converse part, assuming that (8) has a feasible beamforming solution $\mathbf{u}_k, \mathbf{v}_k\ (k \in \mathcal{K})$, we claim that all the clauses in the given 3SAT problem can be satisfied. Denote $p_k = \|\mathbf{v}_k\|^2\ (k \in \mathcal{K})$. For user $k = 1, 2, ..., n$, we have

$$\begin{aligned} 1 &\leq \text{SINR}_k \\ &= |\mathbf{u}_k^\dagger \mathbf{E} \mathbf{v}_k|^2 \\ &\leq \max_{\|\mathbf{u}_k\|=1, \|\mathbf{v}_k\|^2 = p_k} |\mathbf{u}_k^\dagger \mathbf{E} \mathbf{v}_k|^2 \\ &= \max_{\|\mathbf{v}_k\|^2 = p_k} \lambda_{\max}\left(\mathbf{E} \mathbf{v}_k \mathbf{v}_k^\dagger \mathbf{E}^\dagger\right) \\ &= \max_{\|\mathbf{v}_k\|^2 = p_k} \mathbf{v}_k^\dagger \mathbf{E}^\dagger \mathbf{E} \mathbf{v}_k \\ &= 6 p_k, \end{aligned} \tag{12}$$

which gives

$$p_k \geq \frac{1}{6},\ k = 1, 2, ..., n. \tag{13}$$

From (12), we also have

$$|\mathbf{u}_k^\dagger \mathbf{H}_{kk} \mathbf{v}_k|^2 \leq 6,\ k \in \mathcal{K}. \tag{14}$$

Construct $\bar{\mathbf{v}}_k$ from $\mathbf{v}_k = (x_k, y_k)^{\text{T}}$ as follows:

$$\bar{\mathbf{v}}_k = \begin{cases} \mathbf{e}_2^1, & \text{if } |x_k| \geq |y_k|; \\ \mathbf{e}_2^2, & \text{if } |x_k| < |y_k|. \end{cases} \tag{15}$$

We prove by contradiction that the beamforming vectors (15) can make the dimension of the interference subspace at each receiver $k\,(k = n+1, n+2, ..., n+m)$ at most two. Take the previous example and assume that at receiver $n+1$ the dimension of the interference subspace is three, which implies that the interference subspace at receiver $n+1$ with beamformers (15) is made of

$$\bar{\mathcal{I}}_{n+1,1} = M \mathbf{e}_3^1,\ \bar{\mathcal{I}}_{n+1,2} = M \mathbf{e}_3^2,\ \bar{\mathcal{I}}_{n+1,3} = M \mathbf{e}_3^3. \tag{16}$$

Combining (10), (15), and (16), we have

$$|y_1| \geq |x_1|, |x_2| \geq |y_2|, |y_3| \geq |x_3|. \tag{17}$$

Let $p = \min_{1 \leq i \leq n} \{p_i\}$. Consider the SINR at $(n+1)$-th receiver,

$$\begin{aligned} &\text{SINR}_{n+1} \\ &= \frac{|\mathbf{u}_{n+1}^\dagger \mathbf{H}_{n+1,n+1} \mathbf{v}_{n+1}|^2}{1 + |\mathbf{u}_{n+1}^\dagger \mathbf{H}_B \mathbf{v}_1|^2 + |\mathbf{u}_{n+1}^\dagger \mathbf{H}_C \mathbf{v}_2|^2 + |\mathbf{u}_{n+1}^\dagger \mathbf{H}_F \mathbf{v}_3|^2} \\ &\leq \frac{6}{1 + |\mathbf{u}_{n+1}^\dagger \mathbf{H}_B \mathbf{v}_1|^2 + |\mathbf{u}_{n+1}^\dagger \mathbf{H}_C \mathbf{v}_2|^2 + |\mathbf{u}_{n+1}^\dagger \mathbf{H}_F \mathbf{v}_3|^2} \\ &= \frac{6}{1 + |\mathbf{u}_{n+1}^\dagger \mathcal{I}_{n+1,1}|^2 + |\mathbf{u}_{n+1}^\dagger \mathcal{I}_{n+1,2}|^2 + |\mathbf{u}_{n+1}^\dagger \mathcal{I}_{n+1,3}|^2} \\ &\leq \frac{6}{1 + M^2 \frac{p}{2}\left(|\mathbf{u}_{n+1}^\dagger \mathbf{e}_3^1|^2 + |\mathbf{u}_{n+1}^\dagger \mathbf{e}_3^2|^2 + |\mathbf{u}_{n+1}^\dagger \mathbf{e}_3^3|^2\right)} \\ &= \frac{6}{1 + M^2 p / 2} \quad (\|\mathbf{u}_{n+1}\| = 1) \\ &< 1, \left(\text{from } M^2 = 72 \text{ and } (13)\right) \end{aligned}$$

TABLE I
COMPLEXITY STATUS OF THE MAX-MIN FAIRNESS LINEAR TRANSCEIVER DESIGN PROBLEM IN THE MULTI-USER MIMO INTERFERENCE CHANNEL

Rx antennas \ Tx antennas	$N_k = 1$	$N_k = 2$	$N_k \geq 3$
$M_k = 1$	Polynomial Time Solvable	Polynomial Time Solvable	Polynomial Time Solvable
$M_k = 2$	Polynomial Time Solvable	Unknown	Strongly NP-hard
$M_k \geq 3$	Polynomial Time Solvable	Strongly NP-hard	Strongly NP-hard

which contradicts the fact $\text{SINR}_k \geq 1\,(k \in \mathcal{K})$. The first inequality is due to (14) and the second inequality is due to (10) and (17). Furthermore, we can choose $\bar{\mathbf{u}}_k$ such that

$$\bar{\mathbf{u}}_k^\dagger \mathbf{H}_{kj} \bar{\mathbf{v}}_j = 0, \forall\, j \neq k.$$

Actually, (15) gives a truth assignment which satisfies all the clauses in the 3SAT problem as in [13].

Finally, this transformation is in polynomial time. Since the 3SAT problem is NP-complete, we can conclude that the problem of checking whether the given target SINR is feasible is strongly NP-hard. ■

Table I summarizes the complexity status of the max-min fairness linear transceiver design problem in the multi-user MIMO interference channel. The complexity remains unknown for the scenario where all transmitters and receivers are equipped with exactly two antennas ($M_k = N_k = 2, k \in \mathcal{K}$).

IV. A CYCLIC COORDINATE ASCENT ALGORITHM

This section considers the numerical algorithm for the max-min fairness linear transceiver design problem (4). Since the problem is strongly NP-hard (proved in Section III), we aim to develop an efficient algorithm to find a local optimal solution.

Cyclic Coordinate Ascent Algorithm for the Linear Transceiver Design Problem

S1. Initialization: Given $\mathbf{v}^0$ and tolerance ϵ. Set $n = 0$.

S2. Computation: Compute the optimal LMMSE receive beamformer $\mathbf{u}^n \in \phi(\mathbf{v}^n)^a$, where $\phi(\mathbf{v}^n)$ represents the optimal solution set with fixed $\mathbf{v}^n$. Denote the objective value by $G_{2n} \triangleq G(\mathbf{u}^n, \mathbf{v}^n)$.

S3. Computation: Solve

$$\begin{array}{ll} \max & \min\limits_{k \in \mathcal{K}} \left\{ \dfrac{|\,(\mathbf{u}_k^n)^\dagger\, \mathbf{H}_{kk} \mathbf{v}_k|^2}{\sigma_k^2 + \sum_{j \neq k} |\,(\mathbf{u}_k^n)^\dagger\, \mathbf{H}_{kj} \mathbf{v}_j|^2} \right\} \\ \text{s.t.} & \|\mathbf{v}_k\|^2 \leq P_k,\, k \in \mathcal{K} \end{array} \quad (18)$$

to obtain the optimal transmit beamformer $\mathbf{v}^{n+1} \in \psi(\mathbf{u}^n)$ with fixed $\mathbf{u}^n$, where $\psi(\mathbf{u}^n)$ represents the optimal solution set of (18). Denote the objective value by $G_{2n+1} \triangleq G(\mathbf{u}^n, \mathbf{v}^{n+1})$.

S4. Termination:
- if $G_{2n+1} - G_{2n-1} \leq \epsilon$, terminate the algorithm;
- else set $n = n + 1$ and go to **S2**.

[a] Notice that $\phi(\mathbf{v}) = \{\phi_\alpha(\mathbf{v}) : |\alpha| = 1\}$, where $\phi_\alpha(\mathbf{v}) = \alpha\phi_1(\mathbf{v})$, the k-th block of $\phi_1(\mathbf{v})$ is $\tilde{\mathbf{u}}_k/\|\tilde{\mathbf{u}}_k\|$, and $\tilde{\mathbf{u}}_k = \left(\sum_{j=1}^K \mathbf{H}_{kj}\mathbf{v}_j\,(\mathbf{H}_{kj}\mathbf{v}_j)^\dagger + \sigma_k^2 \mathbf{I}\right)^{-1} \mathbf{H}_{kk}\mathbf{v}_k$.

Denote $\mathbf{u}^n = (\mathbf{u}_1^n, ..., \mathbf{u}_K^n)\,, \mathbf{v}^n = (\mathbf{v}_1^n, ..., \mathbf{v}_K^n)\,,$ where $n \geq 0$ denotes the iteration index. A cyclic coordinate ascent algorithm (CCAA) is proposed here to solve the optimization problem (4). Using this technique, problem (4) decomposes into a series of simple subproblems. Each subproblem is related to (less than) half of the design variables and can be solved to global optimality in polynomial time [6].The next result shows that the sequence generated by the proposed CCAA converges to a KKT point of (4).

Theorem 4.1: Either the sequence $\{(\mathbf{u}^n, \mathbf{v}^n)\}$ generated by the cyclic coordinate ascent algorithm terminates at a stationary point or each of its accumulation points $(\bar{\mathbf{u}}, \bar{\mathbf{v}})$ is a stationary point of (4).

Proof: According to CCAA, the sequence $\{G_n\}$ is monotonically increasing. Due to the boundness of $\{||\mathbf{u}^n||\}$, there exists a subsequence $\mathcal{N}$ such that $\mathbf{u}^n \to \bar{\mathbf{u}}, n \in \mathcal{N}$. Since $\left\{\|\mathbf{v}^{n+1}\|\right\}$ is also bounded, it follows (without loss of generality) that $\mathbf{v}^{n+1} \to \bar{\mathbf{v}}, n \in \mathcal{N}$. Consequently, there holds

$$G_{2n+1} \to \bar{G} \triangleq G(\bar{\mathbf{u}}, \bar{\mathbf{v}}), n \in \mathcal{N}.$$

Due to the monotonicity of $\{G_n\}$, we have $G_n \to \bar{G}$.

Next, we claim that $\bar{\mathbf{v}} \in \psi(\bar{\mathbf{u}})$ and $\bar{\mathbf{u}} \in \phi(\bar{\mathbf{v}})$. Because of $\mathbf{v}^{n+1} \in \psi(\mathbf{u}^n)$, for any feasible $\mathbf{v}$, we have

$$G(\mathbf{u}^n, \mathbf{v}) \leq G(\mathbf{u}^n, \mathbf{v}^{n+1}), n \in \mathcal{N}.$$

Taking limits from both sides of the above inequality, it follows

$$G(\bar{\mathbf{u}}, \mathbf{v}) \leq G(\bar{\mathbf{u}}, \bar{\mathbf{v}}),$$

which implies $\bar{\mathbf{v}} \in \psi(\bar{\mathbf{u}})$. Moreover, consider the subsequence $\{G_{2n+2}\}_{n \in \mathcal{N}}$ and its limit point is $\bar{G}$. Since $\phi_1(\mathbf{v})$ is continuous, we have

$$\phi_1(\mathbf{v}^{n+1}) \to \phi_1(\bar{\mathbf{v}}), n \in \mathcal{N},$$

$$G_{2n+2} = G(\phi_1(\mathbf{v}^{n+1}), \mathbf{v}^{n+1}) \to G(\phi_1(\bar{\mathbf{v}}), \bar{\mathbf{v}}) = G(\bar{\mathbf{u}}, \bar{\mathbf{v}}), n \in \mathcal{N}.$$

Thus, $\bar{\mathbf{u}} \in \phi(\bar{\mathbf{v}})$.

Finally, we prove that $(\bar{\mathbf{u}}, \bar{\mathbf{v}})$ is a stationary point. It is clear that $(\bar{\mathbf{u}}, \bar{\mathbf{v}})$ is feasible. The fact $\bar{\mathbf{v}} \in \psi(\bar{\mathbf{u}})$ implies that there exist $\lambda_1, \lambda_2, ..., \lambda_K, \eta_1, \eta_2, ..., \eta_K$ such that

$$\left\{ \begin{array}{l} \sum\limits_{k \in \mathcal{K}} \lambda_k \nabla_{\mathbf{v}_i} \text{SINR}_k(\bar{\mathbf{u}}, \bar{\mathbf{v}}) = 2\mu_i \bar{\mathbf{v}}_i, \forall\, i \in \mathcal{K}, \\ \sum\limits_{k \in \mathcal{K}} \lambda_k = 1, \lambda_k = 0 \text{ if } \text{SINR}_k(\bar{\mathbf{u}}, \bar{\mathbf{v}}) < \bar{G}, \\ \lambda_k \geq 0, \mu_k \geq 0, \mu_k(P_k - \|\mathbf{v}_k\|^2) = 0, \forall\, k \in \mathcal{K}. \end{array} \right. \quad (19)$$

Notice that $\text{SINR}_k(\mathbf{u}, \mathbf{v})$ depends only on $\mathbf{u}_k$ with fixed $\mathbf{v}$, so $\bar{\mathbf{u}} \in \phi(\bar{\mathbf{v}})$ implies that there exist $\eta_1, \eta_2, ..., \eta_K$ satisfying

$$\nabla_{\mathbf{u}_k} \text{SINR}_k(\bar{\mathbf{u}}, \bar{\mathbf{v}}) = 2\eta_k \bar{\mathbf{u}}_k, \forall\, k \in \mathcal{K}. \quad (20)$$

Multiplying (20) by λ_k and combining it with (19) and the feasibility condition of $(\bar{\mathbf{u}}, \bar{\mathbf{v}})$, we obtain the KKT condition

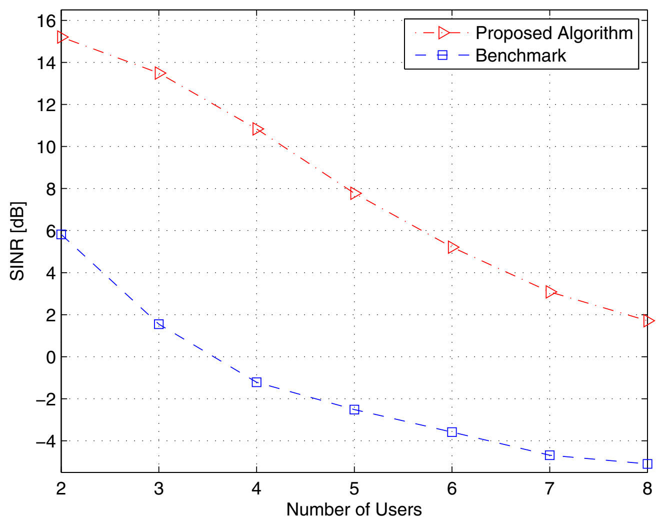

Fig. 1. Performance comparison of the proposed algorithm and channel matched beamformer versus different numbers of users with SNR= 10 dB.

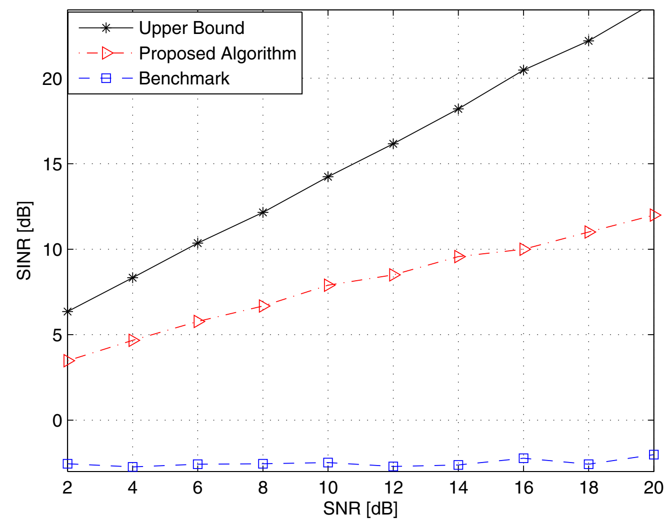

Fig. 2. Performance comparison of the proposed algorithm and channel matched beamformer versus different SNRs with $K = 5$.

of problem (4), where $\mu_k \geq 0$ and $\lambda_k \eta_k$ are the Lagrangian multipliers corresponding to the constraints $P_k - \|\mathbf{v}_k\|^2 \geq 0$ and $\|\mathbf{u}_k\|^2 - 1 = 0$, respectively. Hence $(\bar{\mathbf{u}}, \bar{\mathbf{v}})$ is a stationary point of problem (4). ■

V. Numerical Results

Numerical results are conducted in this section to evaluate the effectiveness of the proposed CCAA. We consider a MIMO multi-user interference channel with three antennas ($N_k = 3$) at each transmitter and two antennas ($M_k = 2$) at each receiver. The channel matrices $\mathbf{H}_{kj}$ [9] are generated obeying the complex normal distribution, i.e., $\text{vec}(\mathbf{H}_{kj}) \sim \mathcal{CN}(\mathbf{0}, \mathbf{I})$. Channel matched beamformers $\tilde{\mathbf{u}}_k$ and $\tilde{\mathbf{v}}_k$ (the left and right singular vector corresponding to the largest singular value of the direct-link channel matrices $\mathbf{H}_{kk}$) are used as the benchmark.

Fig. 1 shows the SINR performance comparison of the proposed algorithm and the benchmark versus different numbers of users, where each plotted point is obtained by averaging over 100 independent channel realizations. As the number of users in the network increases, the interference to each user increases. Therefore, each user's SINR utility decreases. It can be seen from Fig. 1 that the proposed algorithm achieves significant improvement over the benchmark solution. Moreover, most of the improvement is achieved in the first several iterations, making the algorithm attractive in practical implementations.

The SINR performance comparison of the proposed algorithm and the benchmark versus different SNRs is shown in Fig. 2 over 100 channel realizations. The upper bound for the optimal value of linear transceiver design problem (5) is given by $\min_{k \in \mathcal{K}} \left\{ |\tilde{\mathbf{u}}_k^\dagger \mathbf{H}_{kk} \tilde{\mathbf{v}}_k|^2 / \sigma_k^2 \right\}$, which is not achievable in general. However, it still can be used as an ultimate upper bound. We observe from Fig. 2 that the proposed algorithm achieves at least half of the max-min optimality and yields substantially higher minimum SINR than the channel matched beamformer solution.

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