On the Complexity of Leakage Interference Minimization for Interference Alignment

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***Abstract*—For a general MIMO interference channel, we can determine the feasibility of linear interference alignment via minimizing the leakage interference. This paper gives a complete complexity characterization of the leakage interference minimization problem. It is shown that, when each transmitter (receiver) is equipped with at least three antennas and each receiver (transmitter) is equipped with at least two antennas, the problem of checking whether the interference in the network can be perfectly aligned is strongly NP-hard. Moreover, when each transmit/receive node is equipped with two or more antennas, leakage interference minimization can not be solved (even approximately) in polynomial time, unless P = NP.**

Index Terms — Complexity analysis, interference alignment, leakage interference minimization.

I. INTRODUCTION

Interference alignment is shown to achieve maximum spatial degrees of freedom for a multi-user interference network with time-varying/frequency-selective channel coefficients at high SNR [1]. The idea is to appropriately choose the transmit beamforming directions so that the interference signals at each receiver will lie in a low-dimensional interference subspace. In this way, each receiver can successfully extract the desired signal as long as it lies outside the interference subspace. An improved interference alignment scheme is proposed in [2] to achieve a higher multiplexing gain in a single-input single-output (SISO) interference channel for any number of channel realizations.

Assuming perfect channel state information (CSI) is known, an important open problem is to determine the feasibility of linear interference alignment for constant multi-input multi-output (MIMO) interference networks. Reference [3] shows that the interference alignment problem is almost surely feasible for proper systems and almost surely infeasible for improper systems. A constructive proof of achievability of interference alignment is given in [4] for $(N+1)$-user constant MIMO channel with each node equipped with N antennas. Distributed algorithms are proposed in [5], [6], [7] for interference alignment and they appear to work well in small networks. In particular, the leakage interference is minimized in [5] in order to achieve interference alignment.

In this paper, we provide a complete complexity analysis of the leakage interference minimization problem. Let N_k and M_k denote the number of antennas of transmitter k and receiver k, respectively. We show that if $\min\{M_k, N_k\} \geq 2$ and $M_k + N_k \geq 5$ for all k, then the problem of determining if the leakage interference can be made zero for a given degrees of freedom at each receiver is strongly NP-hard. This result extends the complexity results of [8]. Furthermore, we show that the minimization of leakage interference is strongly NP-hard even when $\min_k\{M_k, N_k\} \geq 2$. These results suggest that minimizing leakage interference to achieve interference alignment is computationally challenging when the number of users in the system increases.

We adopt the following notations in this paper. Lowercase boldface and uppercase boldface are used for vectors and matrices. For given matrix $\mathbf{H}$, Rank$(\mathbf{H})$ and $\mathbf{H}^{\mathrm{T}}$ denote the rank and the transpose of $\mathbf{H}$, respectively. $\mathbf{I}_d$ represents the $d \times d$ identity matrix. We use $\mathbf{e}_n^k$ to denote a n-dimensional column vector with its k-th element being one and other elements zero. Finally, we use $\mathcal{K}$ to denote the set of users.

II. PROBLEM FORMULATION

Consider a MIMO interference network with K pairs of users, so we have $\mathcal{K} = \{1, 2, ..., K\}$. Let $\mathbf{H}_{kj} \in \mathbb{R}^{M_k \times N_j}$ represent the channel matrix of real coefficients between transmitter j and receiver k (all the complexity results in this paper can extend to complex channel matrices). Let d_k denote the degrees of freedom achieved by the k-th transmitter-receiver pair.

Interference Alignment. Given d_k $(k \in \mathcal{K})$ and $\mathbf{H}_{kj} \in \mathbb{R}^{M_k \times N_j}$ $(k, j \in \mathcal{K})$, find transmit precoding matrices $\mathbf{V}_k \in \mathbb{R}^{N_k \times d_k}$ $(k \in \mathcal{K})$ and zero-forcing interference suppression matrices $\mathbf{U}_k \in \mathbb{R}^{M_k \times d_k}$ $(k \in \mathcal{K})$ with Rank$(\mathbf{U}_k)$ = Rank$(\mathbf{V}_k) = d_k$ for all $k \in \mathcal{K}$ such that

$$\mathbf{U}_k^{\mathrm{T}}\mathbf{H}_{kj}\mathbf{V}_j = \mathbf{0}_{d_k \times d_j}, \ \forall\, k \neq j; \quad (1)$$

$$\mathrm{Rank}\big(\mathbf{U}_k^{\mathrm{T}}\mathbf{H}_{kk}\mathbf{V}_k\big) = d_k, \ \forall\, k \in \mathcal{K}. \quad (2)$$

Reference [5] explains why condition (2) is automatically satisfied almost surely if the channel matrices have no special structures. Supposing that $\mathbf{U}_k$ $(k \in \mathcal{K})$ and $\mathbf{V}_k$ $(k \in \mathcal{K})$, satisfying condition (1), have been found, condition (2) will be satisfied with probability one if all the elements of channel

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matrices are randomly and independently generated from a continuous distribution. Therefore, we focus on the problem of finding $\mathbf{U}_k$ and $\mathbf{V}_k$ such that

$$\begin{cases} \mathbf{U}_k^{\mathrm{T}}\mathbf{H}_{kj}\mathbf{V}_j = \mathbf{0}_{d_k\times d_j}, \forall\ k\neq j; \\ \mathrm{Rank}(\mathbf{U}_k) = \mathrm{Rank}(\mathbf{V}_k) = d_k, \forall\ k\in\mathcal{K}. \end{cases} \tag{3}$$

Leakage Interference Minimization. An effective method of checking the feasibility of problem (3) and finding its solution is via the leakage interference minimization (LIM) [5], which is given by

$$\begin{aligned} \min\quad & \sum_{k\in\mathcal{K}}\sum_{j\neq k}\|\mathbf{U}_k^{\mathrm{T}}\mathbf{H}_{kj}\mathbf{V}_j\|^2 \\ \text{s.t.}\quad & \mathbf{U}_k^{\mathrm{T}}\mathbf{U}_k = \mathbf{I}_{d_k},\ \mathbf{V}_k^{\mathrm{T}}\mathbf{V}_k = \mathbf{I}_{d_k},\ k\in\mathcal{K}. \end{aligned} \tag{4}$$

The conditions $\mathrm{Rank}(\mathbf{U}_k) = \mathrm{Rank}(\mathbf{V}_k) = d_k$ can be rewritten as $\mathbf{U}_k^{\mathrm{T}}\mathbf{U}_k = \mathbf{V}_k^{\mathrm{T}}\mathbf{V}_k = \mathbf{I}_{d_k}$. The optimal value of (4) is zero if and only if there exists a feasible interference alignment solution to (3).

The decision version of optimization problem (4), denoted by LIM(M), is to decide whether there exists a feasible point at which the objective value of (4) is not greater than M, where M is the given threshold value. The answer to the decision problem is binary, true or false. In particular, a true (or false) answer to problem LIM(0) corresponds to precisely the existence (or non-existence) of a feasible interference alignment solution to problem (3) for the given tuple of degrees of freedom $\mathbf{d} = (d_1, d_2, ..., d_K)^{\mathrm{T}}$.

III. Complexity Analysis

We now investigate the complexity status of problem LIM(M) and its special case LIM(0) for the multi-user MIMO interference network. We refer the interference space $\mathcal{I}_k$ at receiver k to the span of all interference vectors from other users as follows:

$$\mathcal{I}_k = \mathrm{Span}\left\{\bigcup_{j\neq k}\bigcup_{n=1}^{d_j}\mathbf{H}_{kj}\mathbf{V}_j^n\right\},\ k\in\mathcal{K},$$

where $\mathbf{V}_j^n$ is the n-th column of $\mathbf{V}_j$.

In general, to show the NP-hardness of a continuous feasibility (or optimization) problem, we need to transform a known NP-complete discrete problem to it. To facilitate this transformation, it is necessary to induce certain discrete structure to its solutions. In this paper, the 3SAT and MAX-2SAT problems are employed as our NP-complete problem and special $\mathbf{H}_{kj}$ are chosen to guarantee that the optimal transmit beamforming vector $\mathbf{v}_k$ can be only $\mathbf{e}_2^1$ or $\mathbf{e}_2^2$.

Theorem 3.1: Problem LIM(0) is strongly NP-hard when $\min\{M_k, N_k\}\geq 2$ and $M_k + N_k \geq 5, \forall\ k\in\mathcal{K}$. Thus, in this case, the problem of checking the achievability of a given tuple of degrees of freedom, $\mathbf{d} = (d_1, d_2, ..., d_K)^{\mathrm{T}}$, is strongly NP-hard.

Proof: Without loss of generality, we consider the case $M_k = 3$, $N_k = 2$, $k\in\mathcal{K}$. The proof is based on a polynomial time transformation from 3SAT problem, which is NP-complete [9]. The 3SAT problem is described as follows: given m disjunctive clauses (Recall that for a given set of Boolean variables, a literal is defined as either a Boolean variable itself or its negation, while a disjunctive clause refers to a logical expression consisting of the logical "OR" of literals) defined on n Boolean variables such that each clause contains exactly three literals, the question is to check whether there exists a truth assignment for these Boolean variables such that all clauses are satisfied. Specifically, we claim that the problem of checking the minimum value of

$$\begin{aligned} \min\quad & \sum_{k\in\mathcal{K}}\sum_{j\neq k}\left(\mathbf{u}_k^{\mathrm{T}}\mathbf{H}_{kj}\mathbf{v}_j\right)^2 \\ \text{s.t.}\quad & \|\mathbf{u}_k\| = 1,\ \|\mathbf{v}_k\| = 1,\ k\in\mathcal{K} \end{aligned} \tag{5}$$

is zero or not is strongly NP-hard, where $\mathbf{H}_{kj}\in\mathbb{R}^{3\times 2}$ and the given tuple of degree of freedom is $\mathbf{d} = (1, 1, ..., 1)^{\mathrm{T}}$. In this case, all matrices $\mathbf{U}_k$ and $\mathbf{V}_k$ in (4) reduce to column vectors $\mathbf{u}_k$ and $\mathbf{v}_k = (x_k, y_k)^{\mathrm{T}}$, respectively. Intuitively, problem (5) is a special class of homogeneous quartic polynomial minimization problem subject to unit-norm constraints, so it is strongly NP-hard. Next, we show it indeed is.

Given any instance of 3SAT problem consisting of m disjunctive clauses $c_1, c_2, ..., c_m$ defined on n Boolean variables $x_1, x_2, ..., x_n$, we construct a MIMO interference channel with $m+n$ users, where Boolean variable x_i corresponds to the i-th user, including "variable transmitter" i (V-TX$_i$) and "variable receiver" i (V-RX$_i$), and clause c_j corresponds to the $(n+j)$-th user, including "clause transmitter" j (C-TX$_j$) and "clause receiver" j (C-RX$_j$). Hence, $\mathcal{K} = \{1, 2, ..., n+m\}$.

Now, we construct the crosstalk channel matrices $\mathbf{H}_{kj}$ among these $m+n$ users. To this aim, we define

$$\mathbf{H}_A = \mathbf{e}_3^1\left(\mathbf{e}_2^1\right)^{\mathrm{T}},\ \mathbf{H}_B = \mathbf{e}_3^1\left(\mathbf{e}_2^2\right)^{\mathrm{T}},\ \mathbf{H}_C = \mathbf{e}_3^2\left(\mathbf{e}_2^1\right)^{\mathrm{T}},$$
$$\mathbf{H}_D = \mathbf{e}_3^2\left(\mathbf{e}_2^2\right)^{\mathrm{T}},\ \mathbf{H}_E = \mathbf{e}_3^3\left(\mathbf{e}_2^1\right)^{\mathrm{T}},\ \mathbf{H}_F = \mathbf{e}_3^3\left(\mathbf{e}_2^2\right)^{\mathrm{T}}.$$

For user $k = 1, 2, ..., n$, let $\mathbf{H}_{kj} = \mathbf{0},\ \forall\ j\in\mathcal{K}\backslash\{k\}$; and for user $k = n+1, n+2, ..., n+m$, let $\mathbf{H}_{kj} = \mathbf{0}$ except for the following three interference matrices

$$\begin{aligned} \mathbf{H}_{kj} &= \begin{cases} \mathbf{H}_B, & \text{if } \alpha_{\pi(k)} = x_j \text{ for some } j; \\ \mathbf{H}_A, & \text{if } \alpha_{\pi(k)} = \bar{x}_j \text{ for some } j, \end{cases} \\ \mathbf{H}_{kj} &= \begin{cases} \mathbf{H}_D, & \text{if } \beta_{\rho(k)} = x_j \text{ for some } j; \\ \mathbf{H}_C, & \text{if } \beta_{\rho(k)} = \bar{x}_j \text{ for some } j, \end{cases} \\ \mathbf{H}_{kj} &= \begin{cases} \mathbf{H}_F, & \text{if } \gamma_{\tau(k)} = x_j \text{ for some } j; \\ \mathbf{H}_E, & \text{if } \gamma_{\tau(k)} = \bar{x}_j \text{ for some } j, \end{cases} \end{aligned} \tag{6}$$

where $c_{k-n} = \alpha_{\pi(k)}\vee\beta_{\rho(k)}\vee\gamma_{\tau(k)}$, α, β, and γ are taken from $\{x, \bar{x}\}$ and π, ρ, and τ are mappings from $\{n+1, n+2, ..., n+m\}$ to $\{1, 2, ..., n\}$. The constructed special instance of (5) with a total of $m+n$ users is given by

$$\begin{aligned} \min\quad & \sum_{k=n+1}^{n+m}\sum_{f\in\{\pi,\rho,\tau\}}\left(\mathbf{u}_k^{\mathrm{T}}\mathbf{H}_{kf(k)}\mathbf{v}_{f(k)}\right)^2 \\ \text{s.t.}\quad & \|\mathbf{u}_k\| = 1,\ \|\mathbf{v}_k\| = 1,\ k\in\mathcal{K}. \end{aligned} \tag{7}$$

This is because the crosstalk interference matrices between all transmitters and V-RX$_i$ $(i = 1, 2, ..., n)$ are zero, so all of V-RXs are interference-free; while C-RX$_j$ $(j = 1, 2, ..., m)$ indeed suffers from crosstalk interferences from three V-TXs.

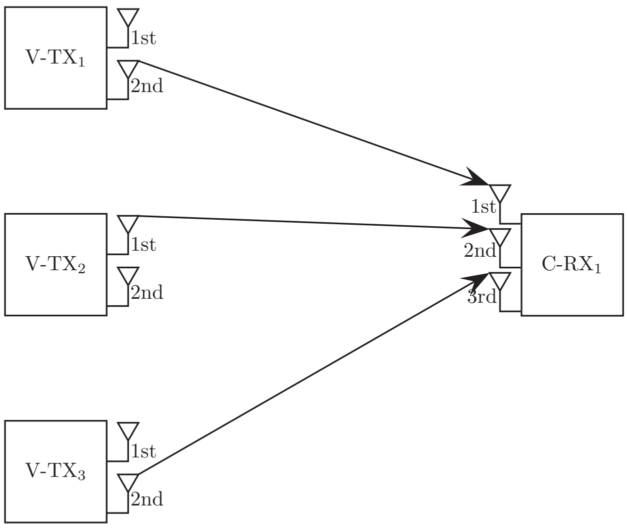

Fig. 1. An illustration of the constructed MIMO interference channel with $c_1 = x_1 \vee \bar{x}_2 \vee x_3$, where all of V-TX$_1$, V-TX$_2$, and V-TX$_3$ are equipped with two antennas while C-RX$_1$ is equipped with three antennas. The solid arrows represent the channel matrices between the corresponding V-TXs and C-RX. According to (6), the channel matrices between V-TX$_1$, V-TX$_2$, and V-TX$_3$ and C-RX$_1$ are $\mathbf{H}_B$, $\mathbf{H}_C$, and $\mathbf{H}_F$, respectively. When the transmitted signal vector at V-TX$_1$ is $\mathbf{v}_1 = (x_1, y_1)^{\mathrm{T}}$, the signal received at C-RX$_1$ will be $y_1\mathbf{e}_3^1$, so we only connect the second antenna of V-TX$_1$ with the first antenna of C-RX$_1$. Similar connections apply to other V-TXs.

Furthermore, problem (7) degenerates into

$$\begin{aligned} \min \quad & \sum_{k=n+1}^{n+m} \sum_{f \in \{\pi,\rho,\tau\}} \left(\mathbf{u}_k^{\mathrm{T}} \mathbf{H}_{kf(k)} \mathbf{v}_{f(k)}\right)^2 \\ \text{s.t.} \quad & \|\mathbf{v}_k\| = 1,\ k = 1,2,...,n, \\ & \|\mathbf{u}_k\| = 1,\ k = n+1, n+2, ..., n+m, \end{aligned} \tag{8}$$

which is a problem of checking whether there exist proper $\mathbf{v}_k$ $(k = 1,2,...,n)$ and $\mathbf{u}_k$ $(k = n+1, n+2, ..., n+m)$ such that all of C-RXs can achieve interference alignment. An example of the constructed MIMO interference channel is depicted in Fig. 1, where $c_1 = x_1 \vee \bar{x}_2 \vee x_3$.

Next, we show that *the 3SAT problem is satisfied if and only if the minimum value of* (8) *is zero.*

If there is a truth assignment to all the Boolean variables $x_1, x_2, ..., x_n$, which makes all of clauses in the 3SAT problem satisfied, we prove that the optimal value of (8) is zero. Let $y_k = 1 - x_k$ $(k = 1,2,...,n)$ and we get a feasible $\mathbf{v}_k$ (either $\mathbf{e}_2^1$ or $\mathbf{e}_2^2$) for problem (8). We claim these beamformers can make the dimension of the interference space at each receiver k $(k = n+1, n+2, ..., n+m)$ at most two. As an illustrative example, suppose that $c_1 = x_1 \vee \bar{x}_2 \vee x_3$ is satisfied. Therefore, at least one of the following facts $x_1 = 1$, $x_2 = 0$, or $x_3 = 1$ hold true. Consider receiver $n+1$. The interference vectors at receiver $n+1$ are as follows:

$$\begin{aligned} \mathbf{H}_{n+1,1}\mathbf{v}_1 &= \mathbf{H}_B\mathbf{v}_1 = y_1\mathbf{e}_3^1, \\ \mathbf{H}_{n+1,2}\mathbf{v}_2 &= \mathbf{H}_C\mathbf{v}_2 = x_2\mathbf{e}_3^2, \\ \mathbf{H}_{n+1,3}\mathbf{v}_3 &= \mathbf{H}_F\mathbf{v}_3 = y_3\mathbf{e}_3^3. \end{aligned}$$

Thus, the dimension of the interference space $\mathcal{I}_{n+1}$ at receiver $n+1$ is at most two and $\mathbf{u}_{n+1}$ can be chosen as its orthogonal complement such that the leakage interference at receiver $n+1$ is zero. The same argument can be used for other users.

Conversely, assuming that the minimum value of (8) is zero, we argue all clauses can be satisfied. Let $\tilde{\mathbf{v}}_k = (\tilde{x}_k, \tilde{y}_k)^{\mathrm{T}}$ $(k = 1,2,...,n)$ be a solution to (8). We make new beamforming vectors $\mathbf{v}_k = (x_k, y_k)^{\mathrm{T}}$ $(k = 1,2,...,n)$ as follows:

$$\mathbf{v}_k = \begin{cases} \mathbf{e}_2^1, & \text{if } \tilde{x}_k \neq 0, \\ \mathbf{e}_2^2, & \text{if } \tilde{x}_k = 0. \end{cases} \tag{9}$$

First, the constructed $\mathbf{v}_k$ is feasible to (8). Furthermore, at each receiver the dimension of the interference space with newly constructed beamforming vectors (9) is not greater than the dimension of the interference space with beamforming vector $\tilde{\mathbf{v}}_k$. As a result, we can choose proper $\mathbf{u}_k$ $(k = n+1, n+2, ..., n+m)$ such that the objective value of (8) at point (9) is zero. Actually, (9) also gives rise to a truth assignment to Boolean variables $x_1, x_2, ..., x_n$. Next, we prove by contradiction that this truth assignment (9) can make all clauses in the 3SAT problem satisfied. Take the previous example. If $c_1 = x_1 \vee \bar{x}_2 \vee x_3$ is not satisfied, it follows that $x_1 = 0$, $x_2 = 1$, and $x_3 = 0$. Due to (9), we have $y_1 = 1$ and $y_3 = 1$, and the interference space at receiver $n+1$ is

$$\mathcal{I}_{n+1} = \text{Span}\{y_1\mathbf{e}_3^1, x_2\mathbf{e}_3^2, y_3\mathbf{e}_3^3\} = \text{Span}\{\mathbf{e}_3^1, \mathbf{e}_3^2, \mathbf{e}_3^3\}.$$

Thus, the dimension of the interference space is three at receiver $n+1$, and the optimal value of (8) is greater than or equal to one, which is a contradiction. As a result, the 3SAT problem is satisfied.

Finally, this transformation is in polynomial time. Since the 3SAT problem is NP-complete, we can conclude that problem LIM(0) is strongly NP-hard.

If letting

$$\mathbf{H}_{kk} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{pmatrix},\ k \in \mathcal{K},$$

we can show the problem of checking the achievability of a given tuple of degrees of freedom is also strongly NP-hard. ■

Theorem 3.1 immediately implies that the problem LIM(M) is strongly NP-hard for any $M \geq 0$ when $\min\{M_k, N_k\} \geq 2$, and $M_k + N_k \geq 5, \forall k \in \mathcal{K}$. Furthermore, it also implies that there does not exist a polynomial time constant factor approximation algorithm for problem LIM(M) unless NP=P. Specially, assume such an algorithm $\mathfrak{A}$ exists. Then for any instance of (5), we can apply $\mathfrak{A}$ to this instance and would obtain a bound I such that $CI \leq I_{\min} \leq I$, where $I_{\min}$ is the optimal value of (5) and $0 < C \leq 1$ is a constant factor. This further implies that $I_{\min} = 0$ if and only if $I = 0$. Consequently, we can use the approximation algorithm $\mathfrak{A}$ to decide whether the minimal value of (5) would be zero or not in polynomial time, contradicting Theorem 3.1.

A relaxation [10] of interference alignment problem (3) is

$$\begin{cases} \sum_{j\neq k} \|\mathbf{U}_k^{\mathrm{T}}\mathbf{H}_{kj}\mathbf{V}_j\|^2 \le \epsilon_k, \forall\, k \in \mathcal{K}; \\ \mathbf{U}_k^{\mathrm{T}}\mathbf{U}_k = \mathbf{I}_{d_k},\ \mathbf{V}_k^{\mathrm{T}}\mathbf{V}_k = \mathbf{I}_{d_k}, \forall\, k \in \mathcal{K}, \end{cases} \tag{10}$$

where $\epsilon_k \ge 0\,(k \in \mathcal{K})$ are the given interference tolerances. Our analysis shows that (10) is still strongly NP-hard.

Corollary 3.1: It is strongly NP-hard to check the feasibility of interference alignment relaxation problem (10) when $\min\{M_k, N_k\} \ge 2$ and $M_k + N_k \ge 5, \forall\, k \in \mathcal{K}$.

Proof: To verify the corollary we need to establish a polynomial time transformation from 3SAT problem to relaxation problem (10). We construct a special instance of problem (10) based on the 3SAT problem as follows:

$$\begin{cases} \sum_{f\in\{\pi,\rho,\tau\}} \left(\mathbf{u}_k^{\mathrm{T}}\mathbf{H}_{kf(k)}\mathbf{v}_{f(k)}\right)^2 \le \epsilon_k, \forall\, k \in \mathcal{K}; \\ \|\mathbf{u}_k\| = 1,\ \|\mathbf{v}_k\| = 1, \forall\, k \in \mathcal{K}, \end{cases} \tag{11}$$

where $\mathbf{H}_{kj}\ (k \neq j \in \mathcal{K})$ are the same as those in the proof of Theorem 3.1 and $0 \le \epsilon_k < 0.5\,(k \in \mathcal{K})$.

We claim that *there exists a truth assignment such that the 3SAT problem is satisfied if and only if there exists a feasible solution to* (11).

The fact that the 3SAT problem is satisfied directly implies that (11) is feasible. For the converse part, we prove that if $\hat{\mathbf{u}}_k$, $\hat{\mathbf{v}}_k = (\hat{x}_k, \hat{y}_k)^{\mathrm{T}}\,(k \in \mathcal{K})$ is feasible to problem (11), then all the clauses in the 3SAT problem can be made satisfiable. Receive beamformer $\hat{\mathbf{u}}_k$ could be $\mathbf{e}_3^1$, $\mathbf{e}_3^2$, or $\mathbf{e}_3^3$. This is because for any choice of $\hat{\mathbf{v}}_k$,

$$\sum_{f\in\{\pi,\rho,\tau\}} \mathbf{H}_{kf(k)}\hat{\mathbf{v}}_{f(k)}\hat{\mathbf{v}}_{f(k)}^{\mathrm{T}}\mathbf{H}_{kf(k)}^{\mathrm{T}},\ k \in \mathcal{K},$$

are always diagonal and the optimal $\hat{\mathbf{u}}_k\ (k \in \mathcal{K})$ is the eigenvector corresponding to the smallest eigenvalue. Construct a group of new transmit-receive beamformers

$$\mathbf{u}_k = \hat{\mathbf{u}}_k,\ \mathbf{v}_k = \begin{cases} \mathbf{e}_2^1, & \text{if } |\hat{x}_k| \ge |\hat{y}_k|; \\ \mathbf{e}_2^2, & \text{if } |\hat{x}_k| < |\hat{y}_k|. \end{cases} \tag{12}$$

We can show

$$\sum_{f\in\{\pi,\rho,\tau\}} \left(\mathbf{u}_k^{\mathrm{T}}\mathbf{H}_{kf(k)}\mathbf{v}_{f(k)}\right)^2 = 0, \forall\, k \in \mathcal{K}, \tag{13}$$

which implies that the 3SAT problem can be made satisfiable from the proof of Theorem 3.1. In fact, assume that (13) is not true, then there exists k_0 and $j_0 \in \{\pi(k_0), \rho(k_0), \tau(k_0)\}$ with

$$\mathbf{u}_{k_0}^{\mathrm{T}}\mathbf{H}_{k_0 j_0}\mathbf{v}_{j_0} = 1. \tag{14}$$

Combining (12) and (14), we have

$$\left(\hat{\mathbf{u}}_{k_0}^{\mathrm{T}}\mathbf{H}_{k_0 j_0}\hat{\mathbf{v}}_{j_0}\right)^2 \ge 0.5.$$

Therefore,

$$\sum_{f\in\{\pi,\rho,\tau\}} \left(\hat{\mathbf{u}}_{k_0}^{\mathrm{T}}\mathbf{H}_{k_0 f(k_0)}\hat{\mathbf{v}}_{f(k_0)}\right)^2 \ge \left(\hat{\mathbf{u}}_{k_0}^{\mathrm{T}}\mathbf{H}_{k_0 j_0}\hat{\mathbf{v}}_{j_0}\right)^2 \ge 0.5,$$

which contradicts the fact that $\hat{\mathbf{u}}_k, \hat{\mathbf{v}}_k\ (k \in \mathcal{K})$ are feasible to (11). ■

For the scenario where each node is equipped with exactly two antennas, we have the following results.

Theorem 3.2: When $\min\{M_k, N_k\} \ge 2, \forall k \in \mathcal{K}$, problem LIM($M$) is strongly NP-hard. However, problem LIM(0) is polynomial time solvable when $M_k = N_k = 2, \forall\, k \in \mathcal{K}$.

Proof: The second part of the theorem is due to [8], where the authors transform the feasibility problem LIM(0) in polynomial time into an instance of 2SAT problem [9]. The 2SAT problem differs from the 3SAT problem in the respect that each of its clauses contain only two literals. Since 2SAT problem is known to be solvable in polynomial time, problem LIM(0) is polynomial time solvable when $M_k = N_k = 2, \forall\, k \in \mathcal{K}$.

Next, we prove the first part of Theorem 3.2. Similar to the proof of Theorem 3.1, we use a polynomial time transformation from the MAX-2SAT problem [9]. The MAX-2SAT problem is, given a set of disjunctive clauses, each with two literals in it, and an integer M, we are asked whether there is a truth assignment that satisfies at least M of the clauses.

Given any instance of the MAX-2SAT problem consisting of m disjunctive clauses $c_1, c_2, ..., c_m$ defined on n Boolean variables $x_1, x_2, ..., x_n$, we construct a $(m+n)$-user MIMO interference channel, where Boolean variable x_i corresponds to the i-th user and clause c_j corresponds to the $(n+j)$-th user, with a set of crosstalk channel matrices $\mathbf{H}_{kj} \in \mathbb{R}^{2\times 2}$. To this end, denote $\mathbf{v}_k = (x_k, y_k)^{\mathrm{T}}\,(k \in \mathcal{K})$ and

$$\mathbf{H}_A = \mathbf{e}_2^1\left(\mathbf{e}_2^1\right)^{\mathrm{T}}, \mathbf{H}_B = \mathbf{e}_2^1\left(\mathbf{e}_2^2\right)^{\mathrm{T}},$$
$$\mathbf{H}_C = \mathbf{e}_2^2\left(\mathbf{e}_2^1\right)^{\mathrm{T}}, \mathbf{H}_D = \mathbf{e}_2^2\left(\mathbf{e}_2^2\right)^{\mathrm{T}}.$$

In particular, for $k = 1, 2, ..., n$, let $\mathbf{H}_{kj} = \mathbf{0}, \ \forall\, j \neq k$; and for $k = n+1, n+2, ..., n+m$ and $1 \le j \le n$, let $\mathbf{H}_{kj} = \mathbf{0}$ except for the following two crosstalk channel matrices

$$\mathbf{H}_{kj} = \begin{cases} \mathbf{H}_B, & \text{if } \alpha_{\pi(k)} = x_j \text{ for some } j; \\ \mathbf{H}_A, & \text{if } \alpha_{\pi(k)} = \bar{x}_j \text{ for some } j, \end{cases}$$

$$\mathbf{H}_{kj} = \begin{cases} \mathbf{H}_D, & \text{if } \beta_{\rho(k)} = x_j \text{ for some } j; \\ \mathbf{H}_C, & \text{if } \beta_{\rho(k)} = \bar{x}_j \text{ for some } j, \end{cases}$$

where $c_{k-n} = \alpha_{\pi(k)} \vee \beta_{\rho(k)}$, α and β are taken from $\{x, \bar{x}\}$ and π and ρ are mappings from $\{n+1, n+2, ..., n+m\}$ to $\{1, 2, ..., n\}$. The constructed problem is

$$\begin{aligned} \min \quad & \sum_{k=n+1}^{n+m} \sum_{f\in\{\pi,\rho\}} \left(\mathbf{u}_k^{\mathrm{T}}\mathbf{H}_{kf(k)}\mathbf{v}_{f(k)}\right)^2 \\ \text{s.t.} \quad & \|\mathbf{v}_k\| = 1,\ k = 1, 2, ..., n, \\ & \|\mathbf{u}_k\| = 1,\ k = n+1, n+2, ..., n+m. \end{aligned} \tag{15}$$

Problem (15) is a special instance of (4). We show that *there exists a truth assignment such that at least M of the clauses are satisfied if and only if the optimal value of* (15) *is less than or equal to $m - M$.*

If there exists a truth assignment such that M of the clauses are satisfied, we prove that the optimal value of (15) is less than or equal to $m - M$. Set $y_k = 1 - x_k\ (k = 1, 2, ..., n)$.

- For a satisfied clause j_0, the interference space $\mathcal{I}_{n+j_0}$ at receiver $n+j_0$ is $\{\mathbf{0}\}$, $\{\mathbf{e}_2^1\}$, or $\{\mathbf{e}_2^2\}$, so we can choose

TABLE I
COMPLEXITY STATUS OF THE PROBLEM LIM(0) IN THE MULTI-USER MIMO INTERFERENCE CHANNEL

Rx antennas \ Tx antennas	$N_k = 1$	$N_k = 2$	$N_k \geq 3$
$M_k = 1$	Polynomial Time Solvable	Polynomial Time Solvable	Poly. Time Solvable
$M_k = 2$	Polynomial Time Solvable	Polynomial Time Solvable [8]	Strongly NP-hard
$M_k \geq 3$	Polynomial Time Solvable	Strongly NP-hard	Strongly NP-hard

$\mathbf{u}_{n+j_0}$ as the orthogonal complement of $\mathcal{I}_{n+j_0}$ to make the leakage interference at receiver $n + j_0$ zero;

- For an unsatisfied clause j_0, the interference space $\mathcal{I}_{n+j_0}$ is Span $\{\mathbf{e}_2^1, \mathbf{e}_2^2\}$, any choice of feasible receive beamforming vector $\mathbf{u}_{n+j_0}$ makes the leakage interference at receiver $n + j_0$ one.

Then the optimal value of (15) is not greater than $m-M$, since we have already found a feasible solution $\mathbf{u}_k, \mathbf{v}_k$ at which the objective value of (15) is $m - M$.

Conversely, assuming that the optimal value of (15) is not greater than $m - M$, we claim at least M of the clauses can be satisfied. Let $\mathbf{v}_k = (x_k, y_k)^{\mathrm{T}}$ and $\mathbf{u}_k$ be the solution to problem (15). For any choice of $\hat{\mathbf{u}}_k$,

$$\sum_{f\in\{\pi,\rho\}} \mathbf{H}_{kf(k)}^{\mathrm{T}} \hat{\mathbf{u}}_{f(k)} \hat{\mathbf{u}}_{f(k)}^{\mathrm{T}} \mathbf{H}_{kf(k)},\ k = n+1, n+2, ..., n+m,$$

are always diagonal and the optimal $\mathbf{v}_k$ in (15) is the eigenvector corresponding to the smallest eigenvalue, it follows that the optimal $\mathbf{v}_k$ should be either $\mathbf{e}_2^1$ or $\mathbf{e}_2^2$. The solution $\{\mathbf{v}_k\}_{k=1}^n$ also gives a truth assignment to the Boolean variables $x_1, x_2, ..., x_n$, and we claim by contradiction that this truth assignment make at least M clauses satisfied. Suppose that less than M clauses are satisfied. From the first part of the arguments, the objective value at newly constructed beamformers $\mathbf{u}_k, \mathbf{v}_k$ would be greater than or equal to $m+1-M$, yielding a contradiction.

Finally, the transformation is in polynomial time. Due to NP-completeness of the MAX-2SAT problem, the leakage interference minimization problem (15) is strongly NP-hard, so is problem LIM(M). ∎

Consider problem LIM(M) in the multi-input single-output (MISO) channel. In this case, problem (4) decomposes into K independent eigenvalue problems:

$$\begin{aligned} \min \quad & \mathrm{Tr}\left(\mathbf{V}_k^{\mathrm{T}}\left(\sum_{j\neq k}\mathbf{H}_{jk}^{\mathrm{T}}\mathbf{H}_{jk}\right)\mathbf{V}_k\right) \\ \text{s.t.} \quad & \mathbf{V}_k^{\mathrm{T}}\mathbf{V}_k = \mathbf{I}_{d_k}. \end{aligned} \tag{16}$$

Therefore, problem LIM(M) is polynomial time solvable in this case, so is problem LIM(0). Similar argument applies to the single-input multi-output (SIMO) interference network.

Table I summarizes the complexity status of problem LIM(0), i.e., the problem of checking whether the network can be made interference-free or not with a given tuple of degrees of freedom in the multi-user MIMO interference channel. It shows that the complexity of the feasibility problem really depends on the number of antennas at transmit/receive nodes. In general, when the number of antennas at each transmit/receive node is at least two, the problem LIM(0) is strongly NP-hard except the scenario where each node is equipped with exactly two antennas.

IV. CONCLUSION

Leakage interference minimization is a useful transceiver design approach for interference mitigation in a multi-user communication system. This paper examines the computational complexity of leakage interference minimization for a multi-user time-invariant MIMO channel. Our analysis suggests that there are significant computational challenges in this approach as the number of users in the system increases. Therefore, when each node in the network is equipped with multiple antennas and the number of users in the system is large, we should focus attention to develop efficient algorithms to find the suboptimal transmit/receive beamforming solution instead of the globally optimal solution.

ACKNOWLEDGEMENT

This work was supported in part by the National Natural Science Foundation, Grant 10831006 and the CAS Grant kjcx-yw-s7-03, in part by the Army Research Office, Grant W911NF-09-1-0279, and in part by the National Science Foundation, Grant CMMI-0726336.

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