ON THE COMPLEXITY OF OPTIMAL COORDINATED DOWNLINK BEAMFORMING

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ABSTRACT

In a cellular wireless system, users located at cell edges often suffer significant out-of-cell interference. In this paper we consider a coordinated beamforming approach whereby multiple base stations jointly optimize their downlink beamforming vectors in order to simultaneously improve the data rates of a given group of cell edge users. Assuming perfect channel knowledge, we formulate this problem as the maximization of a system utility function (which balances user fairness and average user rates), subject to individual power constraints at each base station. We show that, for the single carrier case and when the number of antennas at each base station is at least two, the optimal coordinated beamforming problem is strongly NP-hard for both the harmonic mean utility function and the proportional fairness utility function. For the min-rate utility function, we show that the problem is solvable in polynomial time.

1. INTRODUCTION

In a conventional wireless cellular system, base stations from different cells communicate with their respective remote terminals independently; signal processing is performed on a per-cell basis, while the out-of-cell interference is treated as background noise. This architecture often causes undesirable service outages to users situated near cell edges where the out-of-cell interference can be severe. Since the conventional intra-cell signal processing cannot effectively mitigate the impact of intercell interference, we are led to consider coordinated base station beamforming across multiple cells in order to improve the services to edge users. In this paper, we focus on the downlink scenario where the base stations are equipped with multi-antennas and we assume that the channel state information is known. We consider joint optimal beamforming across multiple base stations to simultaneously improve the data rates of a given group of cell edge users. We formulate this problem as the maximization of a system utility function (which balances user fairness and average user rates), subject to individual power constraints at each base station. We show that, for the single carrier case and when the number of antennas at each base station is at least two, the optimal coordinated beamforming problem is strongly NP-hard for both the harmonic mean utility function and the proportional fairness utility function. For the min-rate utility function, this problem is solvable in polynomial time [7].

1.1. Related Work

Downlink beamforming has been studied extensively in the single cell setup. For the multi-cell interference channel, the reference [1] considered coordinated beamforming for the minimization of total weighted transmitted power across the base stations subject to individual SINR constraints at the remote users. It turns out this problem can be transformed into a second order conic programming. For the maximization of weighted sum rates (or some other utility functions) for an interference channel, the corresponding optimization problem [2] is NP-hard except for the single carrier case. In [3], a distributed beamforming approach is proposed. Some recent work on coordinated beamforming are given in [4, 5].

2. PROBLEM FORMULATION

Consider a cellular system in which there are K base staions each equipped with L transmitter antennas. The K base stations wish to transmit respectively to K mobile receivers each having only a single antenna. Each base station can direct a beam to its intended receiver in such a way that the resulting interference to the other mobile units is small. Consider the single carrier case, and let $\mathbf{h}_{ji} \in \mathbb{C}^L$ denote L-dimensional complex channel vector between base station j and receiver i. Let $\mathbf{v}_i \in \mathbb{C}^L$ denote the beamforming vector used by base station i and u_i is a complex scalar denoting the information signal for user i with $\mathbb{E}|u_i|^2 = 1$. The transmitter vector of jth base station is $\mathbf{v}_j u_j$. Then the mathe-

matical model can be described as

$$y_i = \sum_{j=1}^{K} \mathbf{h}_{ji}^{\dagger} (\mathbf{v}_j u_j) + z_i, \quad i = 1, 2, ..., K, \qquad (1)$$

where y_i is the signal received by user i and z_i is the additive white Gaussian complex noise with variance $\sigma_i^2/2$ on each of its real and imaginary components. It is easy to see that the SINR of each user can be expressed as:

$$\text{SINR}_i = \frac{|\mathbf{h}_{ii}^{\dagger}\mathbf{v}_i|^2}{\sigma_i^2 + \sum_{j \neq i} |\mathbf{h}_{ji}^{\dagger}\mathbf{v}_j|^2}. \qquad (2)$$

Adopting an utility function, we can formulate the optimal coordinated downlink beamforming problem as

$$\begin{aligned}
\max \quad & H(r_1, r_2, ...r_K) \\
\text{s.t.} \quad & r_i = \log\left(1 + \frac{|\mathbf{h}_{ii}^{\dagger}\mathbf{v}_i|^2}{\sigma_i^2 + \sum_{j \neq i} |\mathbf{h}_{ji}^{\dagger}\mathbf{v}_j|^2}\right), \qquad (3) \\
& \|\mathbf{v}_i\|^2 \leq P_i, \quad i = 1, 2, ..., K,
\end{aligned}$$

where P_i denotes the power budget of base station i, and $H(\cdot)$ denotes the system utility function which may be any of the following

- **Sum-rate utility:** $H_1 = \frac{1}{K}\sum_{i=1}^{K} r_i$.
- **Proportional fairness utility:** $H_2 = \left(\prod_{i=1}^{K} r_i\right)^{1/K}$.
- **Harmonic mean utility:** $H_3 = K/\left(\sum_{i=1}^{K} r_i^{-1}\right)$.
- **Min-rate utility:** $H_4 = \min_{1 \leq i \leq K} r_i$.

The above beamforming problem (3) is non-convex.

3. COMPLEXITY ANALYSIS

We now investigate the complexity status of the optimal coordinated downlink beamforming problem (3) under various different choices of system utility functions, and also identify subclasses of the problem that are solvable in polynomial time.

1) Maximization of Sum-Rate Utility: Consider the system utility function $H_1 = \frac{1}{K}\sum_{i=1}^{K} r_i$. When $L = 1$, the original optimization problem (3) becomes the following:

$$\begin{aligned}
\max \quad & \sum_{i=1}^{K} r_i \\
\text{s.t.} \quad & r_i = \log\left(1 + \frac{s_i}{\gamma_i + \sum_{j \neq i} \alpha_{ji} s_j}\right), \qquad (4) \\
& 0 \leq s_i \leq P_i, \quad i = 1, 2, ..., K,
\end{aligned}$$

where $s_i = \|\mathbf{v}_i\|^2$, $\alpha_{ji} = \|\mathbf{h}_{ji}\|^2/\|\mathbf{h}_{ii}\|^2$ and $\gamma_i = \sigma_i^2/\|\mathbf{h}_{ii}\|^2$. Problem (4) is known to be NP-hard [2] and the proof is

based on a polynomial time reduction from the maximum independent set problem.

2) Maximization of Harmonic Mean Utility: We now study the complexity status of the optimal coordinated downlink beamforming problem when the system utility function is the harmonic mean utility.

Theorem 3.1 (Harmonic Mean Utility) *For the harmonic mean utility function $H_3 = K/\left(\sum_{i=1}^{K} r_i^{-1}\right)$, the optimal coordinated downlink beamforming problem can be transformed into a convex optimization problem when $L = 1$, but is strongly NP-hard when $L \geq 2$.*

The first part of Theorem 3.1 was proved in [2]. The NP-hardness proof for $L \geq 2$ is quite involved and we outline its main steps below for the special case when each receiver is equipped with two antennas. The single receive antenna case will be treated in the expanded version of this paper. We first notice that, because of the concavity of the harmonic utility function with respect to each beamforming vector $\mathbf{v}_i$, the maximizing solution must be on the boundary of the feasible solution. In this way, we can constrain the optimal beamforming vectors to be taken from two orthogonal vectors $\mathbf{h}_a$ or $\mathbf{h}_b$.

Lemma 3.1 *The function $(\log(1 + (\sigma^2 + x)^{-1})^{-1}$ is concave in $x \geq 0$ for any $\sigma \neq 0$, and $\mathbf{h}_a = (1, 0)$, $\mathbf{h}_b = (0, 1)$ are the only global optimizers for the optimization problem*

$$\begin{aligned}
\min \quad & (\log(1 + (\sigma^2 + x)^{-1})^{-1} + (\log(1 + (\sigma^2 + y)^{-1})^{-1} \\
\text{s.t.} \quad & x + y = 1 > 0, x \geq 0, y \geq 0.
\end{aligned}$$

The NP-hardness proof of Theorem 3.1 is based on a reduction from a variant of the 3SAT [6] problem. To describe this variant, we need to define the UNANIMITY property and the NAE (stands for "not-all-equal") property of a disjunctive clause.

Definition 3.1 *A disjunctive clause satisfies the UNANIMITY property if all literals in the clause have the same value (whether it is the true or the false value). Otherwise it is said to be satisfied in the NAE sense.*

Clearly, a disjunctive clause must be satisfied if it is to have the NAE property. We define the MAX-UNANIMITY-SAT problem as follows: given a positive integer M and a SAT problem consisting of m disjunctive clauses defined on n boolean variables, we ask whether there exists a truth assignment to the boolean variables such that the number of disjunctive clauses with UNANIMITY property is no less than M. If a MAX-UNANIMITY-SAT problem has two literals in each disjunctive clause, then it is called a MAX-2UNANIMITY-SAT problem. Similarly, we can define the NAE-SAT problem: given an integer number M and a 3-SAT problem, we ask whether at least M of the clauses can

be satisfied in the NAE sense. The NAE-SAT problem is known to be NP-complete [6]. The following lemma holds.

Lemma 3.2 *MAX-2UNANIMITY-SAT is NP-complete.*

Proof We construct a polynomial time reduction from the NAE-SAT problem. It can be checked that MAX-2UNANIMITY-SAT is in the class NP. Given a clause with 3 literals, $c = x \vee y \vee z$, let us construct 6 clauses with 2 literals:

$$R(c) : \ x \vee \bar{y}, \ x \vee \bar{z}, \ y \vee \bar{x}, \ y \vee \bar{z}, \ z \vee \bar{x}, \ z \vee \bar{y}. \quad (5)$$

It can be checked that $R(c)$ has the following properties:
1. The number of clauses in $R(c)$ with all literals having same value is at most 4.
2. The clause c is satisfied in the NAE sense if and only if the number of clauses in $R(c)$ with UNANIMITY property is 4.

Now given any instance ϕ of NAE-SAT, we construct an instance $R(\phi)$ as follows: for each clause $c_i = \alpha \vee \beta \vee \gamma$ of ϕ, we add $R(\phi)$ the six clauses in (5), with x, y, z replaced with α, β, γ. If ϕ has m clauses, then $R(\phi)$ will have $6m$ clauses. Let $M = 4m$. Then properties 1 and 2 imply that at least $M = 4m$ clauses in $R(\phi)$ can be satisfied with UNANIMITY property if and only if ϕ is satisfied in the NAE sense. This reduction is in polynomial time.

Proof of Theorem 3.1: Given any instance of MAX-2UNANIMITY-SAT problem ϕ with clauses c_1, c_2, ..., c_m, variables x_1, x_2, ..., x_n and an integer M, where $c_j = \alpha_{j_1} \vee \alpha_{j_2}$, with $\alpha_{j_1}, \alpha_{j_2}$ taken from $\{x_1, \bar{x}_1, x_2, \bar{x}_2, ..., x_n, \bar{x}_n\}$. Define

$$\mathbf{h}_{j_1} = \begin{cases} \mathbf{h}_a, & \text{if } \alpha_{j_1} = x_i \text{ for some } i \\ \mathbf{h}_b, & \text{if } \alpha_{j_1} = \bar{x}_i \text{ for some } i \end{cases}$$
$$\bar{\mathbf{h}}_{j_1} = \begin{cases} \mathbf{h}_b, & \text{if } \alpha_{j_1} = x_i \text{ for some } i \\ \mathbf{h}_a, & \text{if } \alpha_{j_1} = \bar{x}_i \text{ for some } i \end{cases}$$

The vectors $\mathbf{h}_{j_2}$ and $\bar{\mathbf{h}}_{j_2}$ are defined similarly. We construct below an instance[1] of (3) with a total of $K = 2m + n$ users:

$$
\begin{aligned}
\min \quad & \sum_{i=1}^{n} \frac{1}{r_i} + \sum_{j=1}^{m} \left(\frac{1}{r_{c_{j_1}}} + \frac{1}{r_{c_{j_2}}} \right) \\
\text{s.t.} \quad & r_i = \log\left(1 + M_i \|\mathbf{v}_i\|^2\right), \ \|\mathbf{v}_i\|^2 \leq 1, \ 1 \leq i \leq n, \\
& r_{c_{j_1}} = \log\left(1 + \frac{|\mathbf{h}_{c_j}^\dagger \mathbf{v}_{c_j}|^2}{\sigma^2 + |\mathbf{h}_{j_1}^\dagger \mathbf{v}_{j_1}|^2 + |\mathbf{h}_{j_2}^\dagger \mathbf{v}_{j_2}|^2}\right), \\
& r_{c_{j_2}} = \log\left(1 + \frac{|\mathbf{h}_{c_j}^\dagger \mathbf{v}_{c_j}|^2}{\sigma^2 + |\bar{\mathbf{h}}_{j_1}^\dagger \mathbf{v}_{j_1}|^2 + |\bar{\mathbf{h}}_{j_2}^\dagger \mathbf{v}_{j_2}|^2}\right), \\
& \|\mathbf{v}_{c_j}\|^2 \leq 1, \ j = 1, 2, ..., m.
\end{aligned}
$$
$$(6)$$

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[1] Strictly speaking, this construction requires two antennas per receiver since there is a signal power term $\|\mathbf{v}_i\|^2$ (norm squared) in the definition of r_i, which is written as the sum of two squared components ($\mathbf{v}_i$ is two dimensional), not a single squared term as in (3).

where $\sigma^2 = 1$, M_i is a small positive number (to be specified later), and $\mathbf{h}_{c_j}$ is a unit-norm vector $\|\mathbf{h}_{c_j}\| = 1$. Notice that there are m variables $\mathbf{v}_{c_j}$, one per clause in ϕ. In addition, there are n variables $\mathbf{v}_i$, one per boolean variable x_i. Since each variable $\mathbf{v}_{c_j}$ appears only once in $r_{c_{j_1}}$ and $r_{c_{j_2}}$, it follows that $\mathbf{v}_{c_j} = \mathbf{h}_{c_j}$ at global minimum. The correspondence between MAX-2UNANIMITY-SAT problem and the coordinated optimal downlink beamforming problem (6) is listed in Table 1. Note that $r_{c_{j_1}}$ can be obtained from clause c_j according to Table 1 and $r_{c_{j_2}}$ can be obtained from $r_{c_{j_1}}$ by swapping $\mathbf{h}_a$ with $\mathbf{h}_b$. We claim that the num-

Table 1. Variable Correspondence

MAX-2UNANIMITY-SAT	problem (6)
variable x_i	vector $\mathbf{v}_i$
clause c_j	rates $r_{c_{j_1}}$ and $r_{c_{j_2}}$
literal x_i	$|\mathbf{h}_a^\dagger \mathbf{v}_i|^2$
literal $\bar{x}_i$	$|\mathbf{h}_b^\dagger \mathbf{v}_i|^2$

ber of clauses in ϕ satisfied in the UNANIMITY sense is at least M if and only if the minimum of (6) is no greater than

$$
\begin{aligned}
& M\left(1/\log(2) + 1/\log(4/3)\right) \\
& + (m - M)\left(2/\log(3/2)\right) + \sum_{i=1}^{n} 1/\log(1 + M_i).
\end{aligned}
$$
$$(7)$$

The above reduction is in polynomial time. We prove that for sufficiently small M_i, say $M_i = 1/(144m)$, $i = 1, 2, ..., n$, each global minimizer $\mathbf{v}_1^*, \mathbf{v}_2^*, ..., \mathbf{v}_K^*$ of (6) must have unit-norm $\|\mathbf{v}_i^*\| = 1$, $i = 1, 2, ..., n$. To argue $\|\mathbf{v}_1^*\| = 1$, consider the optimization problem with only $\mathbf{v}_1$ as variable and with the constraint $\|\mathbf{v}_1\| \leq 1$ dropped:

$$
\begin{aligned}
\min \quad & 1/\log\left(1 + M_1 \|\mathbf{v}_1\|^2\right) \\
& + \sum_{j \in S_1} 1/\log\left(1 + 1/(\sigma_{j_2}^2 + v_{11}^2)\right) \\
& + \sum_{j \in S_1} 1/\log\left(1 + 1/(\bar{\sigma}_{j_2}^2 + v_{12}^2)\right)
\end{aligned}
$$
$$(8)$$

where $S_1 = \{j \mid x_1 \in c_j\} \bigcup \{j \mid \bar{x}_1 \in c_j\}$, $\sigma_{j_2}^2 = 1 + |\mathbf{h}_{j_2}^\dagger \mathbf{v}_{j_2}|^2$, $\bar{\sigma}_{j_2}^2 = 1 + |\bar{\mathbf{h}}_{j_2}^\dagger \mathbf{v}_{j_2}|^2$, assuming $c_j = x_1 \vee y_{j_2}$ and $j \in S_1$. Denote $\mathbf{v}_1 = (v_{11}, v_{12})^T$. The necessary optimality condition of (8) is

$$
\begin{aligned}
& \frac{-2M_1}{\log(1 + M_1\|\mathbf{v}_1\|^2)^2(1 + M_1\|\mathbf{v}_1\|^2)}\mathbf{v}_1 \\
& + \begin{pmatrix} \displaystyle\sum_{j \in S_1} \frac{2v_{11}/(\sigma_{j_2}^2 + v_{11}^2 + 1)/(\sigma_{j_2}^2 + v_{11}^2)}{\log\left(1 + 1/\left(\sigma_{j_2}^2 + v_{11}^2\right)\right)^2} \\ \displaystyle\sum_{j \in S_1} \frac{2v_{12}/(\bar{\sigma}_{j_2}^2 + v_{12}^2 + 1)/(\bar{\sigma}_{j_2}^2 + v_{12}^2)}{\log\left(1 + 1/\left(\bar{\sigma}_{j_2}^2 + v_{12}^2\right)\right)^2} \end{pmatrix} \\
= \ & 0.
\end{aligned}
$$
$$(9)$$

Table 2. COMPLEXITY STATUS OF OPTIMAL COORDINATED DOWNLINK BEAMFORMING

Problem \ Utility	Sum-Rate	Proportional Fairness	Harmonic Mean	Min-Rate
$L = 1$, any K	Strongly NP-hard [2]	Convex [2]	Convex [2]	Poly. time Algorithm [2, 7]
$L \geq 2$, any K	Strongly NP-hard	Strongly NP-hard	Strongly NP-hard	Poly. time Algorithm [2, 7]

It can be checked that if $M_1 \leq 1/(144m)$, (9) will never hold. Thus, when $M_i \leq 1/(144m)$, the minimizers of (6) must satisfy $\|\mathbf{v}_i^*\| = 1$.

Next, we prove the equivalence of MAX-2UNANIMITY-SAT and (6). In fact, as the solution has property $\|\mathbf{v}_i^*\| = 1, i = 1, 2, ..., n$, (6) can be transformed into

$$
\begin{aligned}
\min \quad & \sum_{j=1}^{m} 1/\log\left(1 + \frac{1}{1 + y_{j_1} + y_{j_2}}\right) \\
& + \sum_{j=1}^{m} 1/\log\left(1 + \frac{1}{1 + z_{j_1} + z_{j_2}}\right) \\
\text{s.t.} \quad & y_i + z_i = 1,\ y_i \geq 0,\ z_i \geq 0,\ \forall i.
\end{aligned}
\tag{10}
$$

where $c_j = \alpha_{j_1} \vee \alpha_{j_2}$ defines the indices j_1 and j_2, and $y_{j_\ell} = |\mathbf{h}_{j_\ell}^\dagger \mathbf{v}_{j_\ell}|^2$ and $z_{j_\ell} = |\bar{\mathbf{h}}_{j_\ell}^\dagger \mathbf{v}_{j_\ell}|^2$, $\ell = 1, 2$. From Lemma (3.1), the objective of (10) is concave for each pair of (y_i, z_i), it follows that the minimizer of (10) must be at $(0, 1)$ or $(1, 0)$ for each pair of (y_i, z_i). Substituting this form (y_i, z_i) into the objective function, we see that the minimum of (10) is $M\left(1/\log(2) + 1/\log(4/3)\right) + (m - M)\left(2/\log(3/2)\right)$, where M is the largest number of clauses with UNANIMITY property (i.e., either $y_{j_1} = y_{j_2} = 0$ or $z_{j_1} = z_{j_2} = 0$). Therefore, the number of clauses in ϕ which is satisfied in the UNANIMITY sense is greater than or equal to M if and only if (7) holds. This completes the polynomial reduction from MAX-2UNANIMITY-SAT problem to the optimal coordinated beamforming problem (3) with the harmonic mean utility function.

3) Maximization of Proportional Fairness Utility: Like the harmonic mean utility, we have the following hardness result.

Theorem 3.2 (Proportional Fairness Utility) *For the proportional fairness utility function* $H_2 = \left(\prod_{i=1}^{K} r_i\right)^{1/K}$, *the optimal coordinated downlink beamforming problem can be transformed into a convex optimization problem when $L = 1$ and is strongly NP-hard when $L \geq 2$.*

The first part of Theorem 3.2 is proved in [2]. For the second part, the argument is similar to that of Theorem 3.1. For space reasons, we omit the details.

4) Maximization of Min-Rate Utility: Let the system utility function be given by $H = H_4$. In this case, the problem can be solved in polynomial time for arbitrary L and

K [7]. In fact the problem (3) becomes

$$
\begin{aligned}
\max \quad & r \\
\text{s.t.} \quad & r \leq \log\left(1 + \frac{|\mathbf{h}_{ii}^\dagger \mathbf{v}_i|^2}{\sigma_i^2 + \sum_{j \neq i} |\mathbf{h}_{ji}^\dagger \mathbf{v}_j|^2}\right), \\
& \|\mathbf{v}_i\|^2 \leq P_i, \quad i = 1, 2, ..., K,
\end{aligned}
\tag{11}
$$

Given a $\bar{r} \geq 0$, we can efficiently check if there exists $\mathbf{v}_i$ such that the constraints in (11) are satisfied. A bisection technique [7] with each step solving a second order cone feasibility problem is proposed for (11).

4. REFERENCES

[1] H. Dahrouj and W. Yu, "Coordinated beamforming for the Multi-Cell Multi-Antenna Wireless System," in *Proc. CISS*, Princeton, NJ, March 2008.

[2] Z.-Q. Luo and S. Zhang, "Dynamic spectrum management: complexity and duality," *IEEE. Journal of Selected Topics in Signal Processing*, Special Issue on Signal Prccessing and Networking for Dynamic Sprectrum Acess, Vol. 2, No. 1, pp. 57-73, Feb. 2008.

[3] R. Zakhour and D. Gesbert, "Coordination on the MISO interference channel using the virtual SINR framework," in *Proc. WSA*'09, 2009.

[4] E. A. Jorswieck, E. G. Larsson and D. Danev, "Complete characterization of the pareto boundary for the MISO interference channel," in *Proc. IEEE Trans. Signal Processing*, Aug. 2008.

[5] E. Bjornson, R. Zakhour, D. Gesbert, and B. Ottersten, "Cooperative multicell precoding: Rate region characterization and distributed strategies with instantaneous and statistical CSI," *IEEE Trans. Signal Process.*, submitted for publication.

[6] C. H. Papadimitriou, *Computational Complexity*, Beijing, China: Tsinghua University Press, 2004.

[7] A. Wiesel, Y. C. Eldar, and S. S. Shitz, "Linear precoding via conic programming for fixed MIMO receivers," *IEEE Trans. Signal Processing*, Vol. 54, no. 1, pp. 161-176, Jan. 2006.