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STABILITY AND ALLOCATION IN A THREE-PLAYER GAME

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We study a three-player cooperative game with transferable utility where the players may form different coalition structures. A new concept of stability of a coalition is introduced, and the existence of a stable coalition is proven. Based on this stability concept, a novel approach is given to determine sensible allocations in a grand coalition of three players. We also compare our result with classical core solution and implement our theory on a specific price model.

Keywords: Stability; cooperative game; three-player game.

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1. Introduction

1.1. *Motivation*

A cooperative game is a game where players can write and enforce binding agreements. A transferable utility (TU) cooperative game in coalitional form is described by a set of players N, called the grand coalition, and a characteristic function $v : 2^N \rightarrow \Re$, which specifies the value of each possible coalition that can be generated if the players in the coalition enforce cooperative behavior. One of the main issues in such a game is how the value should be allocated among the players. Different solution concepts such as the Shapley value, the nucleolus and the core, etc., were introduced in the literature and have been successfully applied in a variety of applications.

The concept of core is formally introduced in Gillies (1959). For a given cooperative game with a grand coalition N consisting of n players and a characteristic function $v(S)$ for each subset S of N, the core is a set of value allocation vectors specified by

$$C = \left\{ x \in \mathbb{R}^n : \sum_{i\in N} x_i = v(N); \sum_{i\in S} x_i \geq v(S),\ \forall S \subseteq N \right\}, \tag{1}$$

where N is the grand coalition with n players. A vector in the core basically says that based on this value allocation, no subset of players has incentive to deviate from the grand coalition, i.e., the grand coalition is stable, if no further actions will be taken when the grand coalition is broken. That is, if we only consider one-step coalition move, the players deviating from the grand coalition will not improve their status quo. It is also well-known that a vector in the core is fair in the sense that no coalition will be sacrificed to subsidize the players out of the coalition.

There are three issues with the concept of core. First, the core can be empty. In this situation, other solutions like Shapley value in Lloyd Shapley (1953), strong ϵ-core in Shapely and Shubik (1966), the kernel in Davis and Maschler (1965), the nucleolus in Schmeidler (1969), or concepts originated from these can be used. However, these concepts do not prevent players from deviating from the proposed allocations. For instance, for a proposed allocation vector x, if $\sum_{i\in S} x_i < v(S)$, then the coalition S may have incentive to deviate from this proposed allocation.

Second, the core is only a myopic stability concept even if nonempty. That is, it only says that for a given value allocation in the core, any coalition will not be better off if it deviates from the grand coalition and no further chain effects are considered. However, if we take into account the possible further reaction of players to the breakdown of the grand coalition, a player or a coalition may have incentive to deviate from the grand coalition even if the proposed allocation vector is in the core. To make this point concrete, let's look at a simple example with three players.

Example 1. Let $N = 1, 2, 3$ and the characteristic function is specified as

$$v(\{1\}) = v(\{2\}) = v(\{3\}) = 1, \quad v(\{2,3\}) = v(\{1,3\}) = 3,$$
$$v(\{1,2\}) = 5, \quad v(\{1,2,3\}) = 7.$$

It is easy to check that the allocation vector $x = (1.2, 4.9, 1.9)$ belongs to the core. Even though it appears that Player 1 would be worse off if it deviates from the grand coalition, we argue that Player 1 has incentive to do so. Assume that Player 1 leaves the grand coalition. Under this assumption, as long as Player 2 receives a payoff no less than 2, the highest payoff achievable if it stands alone or forms a coalition with Player 3, Player 2 would form a coalition with Player 1. On the other hand, Player 2 cannot propose a payoff less than 2 Player 1, otherwise Player 3 can jump in and form a coalition with Player 1 with slightly higher payoffs for both players. Thus, Player 1 can guarantee a payoff at least 2, which justify its deviation from the grand coalition at the first place.

Third, an allocation not in the core for players in the grand coalition does not mean that the grand coalition will break up. We illustrate this point by the following example.

Example 2. Let $N = 1, 2, 3$ and the characteristic function is specified as

$$v(\{1\}) = v(\{2\}) = v(\{3\}) = 1,$$

$$v(\{1,3\}) = v(\{2,3\}) = 5,$$

$$v(\{1,2\}) = 6, \quad v(\{1,2,3\}) = 8.$$

It is easy to check that the allocation vector $x = (3, 3.5, 1.5)$ does not belong to the core. It seems that Players 1 and 3 have incentive to deviate from the grand coalition and form a coalition to increase their total payoff. However, once the grand coalition is broken, as we show later on, Players 1 and 2 would form a coalition and leave Player 3 behind. In this case, the payoff of Player 3 is 1. It is less than its original payoff and thus Player 3 has no incentive to deviate from the grand coalition at the first place.

The above examples demonstrate that after a coalition S decides to break away from the grand coalition, all players will reevaluate their options and involve in a possibly infinite round of coalition formation process. As long as the eventual payoffs of the players in S are better off than those specified by the grand coalition, the players in S have incentive to break away. On the one hand, the condition $\sum_{i\in S} x_i \geq v(S)$ does not necessarily prevent coalition S to leave the grand coalition. On the other hand, the condition $\sum_{i\in S} x_i < v(S)$ does not necessarily mean coalition S will deviate from the grand coalition for sure.

The objective of this paper is to identify allocations of the grand coalition that are appropriate in the sense that given such an allocation, all players have no incentive to deviate from the grand coalition even taking into account the possible future reactions from the players. For this purpose, we first propose a stability concept of a coalition with two players in games with three players. Basically, a coalition is stable if the payoff of this coalition is sufficiently allocable among its players and it is resistant to other players' offers to destroy the coalition. Under this stability concept, we first examine the existence of such a stable coalition. We then use this stability concept to analyze the allocation in a grand coalition. We compare our allocation result with the core and find an interesting relationship.

Our approach works not only for cooperative games in coalitional form, which is reasonable if for any given coalition, players outside the coalition have no impact on the outcome of the coalition, but also for games with externality in the sense that players outside a coalition can exert effort to affect the possible outcome of the coalition. In the latter case, it becomes a challenging issue how to define the characteristic function and different outcomes exist depending on the coalition structures of the players. Because our approach explicitly takes into account different coalition structures, we are able to handle games for which characteristic functions are difficult to define.

To make things simple and easy to understand, we only consider games with three players in this paper. This is the simplest model that contains different coalition structures. Moreover, we assume perfect information in the game.

Despite simple, this model is still very meaningful since there are many examples in real life business in which the market often has three main competitive factions. Each faction itself can also be a coalition of different companies, But when coalition among different factions is concerned, the cooperation inside is tight enough so that we can treat them as one single player. Here we give some real examples of this type of cooperation/competitions:

Internet market. There are three major companies: Google, Microsoft and Yahoo. Although their main products differ from each other in early years, their competitions recently extend to every aspect, including online search, news, map, E-mail and even mobile web-surfing.

In 2008, Microsoft launched an acquisition on Yahoo. However, with the intervention of Google, the acquisition was failed. It was also believed that some level of coalition is formed between Yahoo and Google.

In 2010, Microsoft launched another purchase of Yahoo's searching technique and market by one billion dollar. This time a deal is made and Yahoo's searching engine merged to Microsoft's Bing. Yahoo's focus changes to advertise marketing. In the following five years, 88% of the profit from online search by Microsoft will be paid to Yahoo. A coalition is hence formed to compete with Google's dominance on searching market.

Smart phone OS. The market is almost completely divided by Apple's iPhone and Google's Android. Other competitors like Symbian are almost forced out of the market. Recently a competitive player emerges. A coalition between Nokia and Microsoft is trying to challenge current balance using their big cellphone consumer group and experience of OS design. This example is different from the previous one. A three-player model is formed when a two-player balance is challenged by a new-emerging competitive player.

Our stability approach considers coalition moves beyond one step. Thus, it is appropriate to compare a farsighted stability concept called consistency introduced in Chwe (1994). A set of coalition structures referred to as states is called consistent if for any state in the set and for arbitrary one-step coalition move, the new state is indirectly dominated by another state in the set and at least one participant among players initiating the one-step coalition move is worse off in the final state than the original state and thus the deviation will not happen since there exists at least one participant who concerns an unfavored final state. In Chwe (1994), a one-step coalition move is allowed only if the players initiating the move are better off immediately after the move. However, we allow one-step coalition moves only if the players initiating the moves are ultimately better off. It is possible that those players are worse off immediately after the one-step moves. In addition, unlike Chwe (1994) which focuses on defining and identifying all possible stable coalition structures, we are mainly concerned about conditions under which allocations of the grand coalition are reasonable in the sense that each player is satisfied, i.e., he cannot guarantee a larger payoff if he breaks away from the grand coalition.

1.2. *Model setting*

Assume there are three players: Players 1, 2 and 3. Each player i can choose strategy S_i from a feasible strategy set $\mathcal{S}_i$. For every strategy S_1, S_2, S_3 chosen by the players, there is a payoff function for each player dnoted as $P_1(S_1, S_2, S_3), P_2(S_1, S_2, S_3)$ and $P_3(S_1, S_2, S_3)$. Let (S_1^N, S_2^N, S_3^N) be the Nash equilibrium when no coalition exists, which means

$$P_1^N := P_1(S_1^N, S_2^N, S_3^N) \geq P_1(S_1, S_2^N, S_3^N), \tag{2}$$

$$P_2^N := P_2(S_1^N, S_2^N, S_3^N) \geq P_2(S_1^N, S_2, S_3^N), \tag{3}$$

$$P_3^N := P_3(S_1^N, S_2^N, S_3^N) \geq P_3(S_1^N, S_2^N, S_3), \tag{4}$$

for all feasible $S_i \in \mathcal{S}_i, i = 1, 2, 3$. To avoid technical difficulty that may obscure our main message, we assume that the Nash equilibria of any noncooperative game involved in this paper exist and give the same payoffs if multiple equilibria exist.

If there is a coalition other than the grand coalition, then it is between two players, say Player 1 and Player 2. The two players can choose their strategies collaboratively hence can be seen as one player in the game. Denote the new Nash equilibrium as $((S_1^*, S_2^*), S_3^*)$, which means for all feasible $S_1 \in \mathcal{S}_1, S_2 \in \mathcal{S}_2$,

$$P_1(S_1^*, S_2^*, S_3^*) + P_2(S_1^*, S_2^*, S_3^*) \geq P_1(S_1, S_2, S_3^*) + P_2(S_1, S_2, S_3^*) \tag{5}$$

and for all feasible $S_3 \in \mathcal{S}_3$,

$$P_3(S_1^*, S_2^*, S_3^*) \geq P_3(S_1^*, S_2^*, S_3). \tag{6}$$

Notice that strategy (S_1^N, S_2^N) is still feasible as a collaborative strategy of the two players, and strategy of Player 3 will still be S_3^N consequently. If the new Nash equilibrium turns out to provide less payoff to Players 1 and 2, i.e.,

$$P_1(S_1^*, S_2^*, S_3^*) + P_2(S_1^*, S_2^*, S_3^*) < P_1^N + P_2^N, \tag{7}$$

they will have the option to choose the old Nash equilibrium as their collaborative strategy. We denote the optimal total payoff of the coalition as

$$\pi_{12} := \max(P_1(S_1^*, S_2^*, S_3^*) + P_2(S_1^*, S_2^*, S_3^*), P_1^N + P_2^N) \tag{8}$$

and the payoff of the left-behind player

$$\pi_3 := \begin{cases} P_3^N & P_1(S_1^*, S_2^*, S_3^*) + P_2(S_1^*, S_2^*, S_3^*) < P_1^N + P_2^N \\ P_3(S_1^*, S_2^*, S_3^*) & \text{Otherwise} \end{cases}. \tag{9}$$

Under this assumption, total payoff for Players 1 and 2 will not decrease after the coalition forms. Hence, they are willing to form a coalition. Consequently, if there exists no coalition, any two players have incentive to cooperate since a coalition can only make things better.

Symmetrically, we can define π_{23}, π_1 in a coalition of Players 2 and 3, and π_{13}, π_2 in a coalition of Players 1 and 3. We assume that all these information is known clearly by all the players.

For a TU game in coalitional form in which there is no externality, we can simply set $\pi_S = v(S)$ for $S \subset \{1, 2, 3\}$ with cardinality 1 or 2.

1.3. *Organization of this paper*

In Sec. 2, we introduce the concept of stability and prove the existence of a stable sub-coalition. Based on this, we analyze the allocation in a grand coalition in Sec. 3. In this section, we also compare our solution with the core. Finally, we apply our general result on a specific price model in Sec. 4 and draw some conclusions in Sec. 5.

2. Stable Coalitions

In this section, we only concentrate on coalition structures in which two players form a coalition and the remaining player stands alone. There are total three such coalition structures. We also assume that payoff cannot be transfered between the coalition and the remaining player. Therefore, the left-behind player's payoff is fixed, however he can propose to form a new coalition by offering to either participant of the current coalition. The two participants of the coalition can discuss the division of the coalition payoff. They can demand a larger allocation in the coalition by threatening to quit current coalition and accept an offer from the remaining player. Notice that we allow the allocation to be dynamically adjusted according to the remaining player's offer, and the only criterion whether a player choose to take an action or not is that his payoff after the move is larger than current one. Base on these facts, we will propose the requirements for the stability of a coalition structure and then analyze the allocation in a stable coalition.

2.1. *Stability*

Before we propose our stability concept, we introduce several definitions. Without loss of generality, we focus on the coalition structure in which Players 1 and 2 form a coalition and Player 3 stands alone.

Denote the allocation of payoff in a coalition between Players 1 and 2 as π_{12}^1 and π_{12}^2, where $\pi_{12}^1 + \pi_{12}^2 = \pi_{12}$.

Definition 3. The coalition of Players 1 and 2 is **allocable** if

$$\pi_{12} \geq \pi_1 + \pi_2. \tag{10}$$

Remark. (1) That a coalition is allocable suggests that there exists some allocation which keeps the players from quitting the coalition (to enlarge his payoff). If the coalition of Players 1 and 2 is not allocable, then for any allocation π_{12}^1, π_{12}^2, at least one player earns less than his left-behind payoff. For example: if $\pi_{12}^1 < \pi_1$, Player 1 may want to quit the current coalition and he would then earn a better payoff after Players 2 and 3 form a coalition.

(2) Notice that we have $\pi_{12} \geq P_1^N + P_2^N$, which means if the coalition of Players 1 and 2 is not allocable, then $\pi_1 > P_1^N$ or/and $\pi_2 > P_2^N$, i.e., at least one player earns more than the payoff of Nash equilibrium if he is left behind.

(3) For a TU game in a coalitional form, that a coalition is allocable simply implies that cooperation generates a total payoff no less than the one generated when the players stand-alone. If the game is superadditive, the coalition of Players 1 and 2 is automatically allocable.

Definition 4. The coalition of Players 1 and 2 is **resistant** if

$$\pi_{12} \geq \pi_{13} - \pi_3 + \pi_2, \tag{11}$$

$$\pi_{12} \geq \pi_{23} - \pi_3 + \pi_1. \tag{12}$$

Remark. That the coalition between Players 1 and 2 is resistant means that it can resist Player 3's offering. Equation (11) implies that if Player 3 offers an allocation of $\pi_{13} - \pi_3$ to Player 1 to allure Player 1 to form a coalition with him (Player 3 will at most offer such an allocation, otherwise he will earn less than his current payoff), Player 2 can offer an equal allocation in current coalition to Player 1 to keep him staying and the allocation left to himself is still no less than π_2, which makes him willing to maintain the coalition. The same logic applies to (12), which indicates that Player 3 cannot allure Player 2 to form a coalition with him.

Now we can give the stability concept of a coalition.

Definition 5. The coalition of Players 1 and 2 is **stable** if it is allocable and resistant.

Remark. That a coalition is stable means both players are happy to stay in the coalition and the left-behind player is not powerful enough to break it.

We also define a stronger stable concept, which will be useful for the analysis of value allocations in the stable coalition.

Definition 6. If

$$\pi_{12} \geq \max(\pi_1, \pi_{13} - \pi_3) + \max(\pi_2, \pi_{23} - \pi_3), \tag{13}$$

the coalition of Players 1 and 2 is called **strongly stable**.

Remark. Notice that (13) is satisfied if and only if

$$\pi_{12} \geq \pi_1 + \pi_2, \tag{14}$$

$$\pi_{12} \geq \pi_1 + (\pi_{23} - \pi_3), \tag{15}$$

$$\pi_{12} \geq (\pi_{13} - \pi_3) + \pi_2, \tag{16}$$

$$\pi_{12} \geq (\pi_{13} - \pi_3) + (\pi_{23} - \pi_3). \tag{17}$$

(14)–(16) are the allocable and resistant conditions for the coalition between Players 1 and 2 to be stable. Hence we have following theorem:

Theorem 7. *Coalition of Players* 1 *and* 2 *is strongly stable if and only if it is stable and* (17) *is satisfied.*

Theorem 8 (Existence of a stable coalition). *If there exists at least one allocable coalition, then there exists at least one stable coalition.*

Proof. To prove the theorem, we first give two straightforward corollaries.

(a) The coalition of Players 1 and 2 is allocable if and only if

$$\pi_{12} + \pi_3 \geq \pi_1 + \pi_2 + \pi_3. \tag{18}$$

(b) The coalition of Players 1 and 2 is resistant if and only if

$$\pi_{12} + \pi_3 \geq \pi_{13} + \pi_2, \tag{19}$$

$$\pi_{12} + \pi_3 \geq \pi_{23} + \pi_1. \tag{20}$$

Without loss of generality, we can assume that the structure with a coalition of Players 1 and 2 generates the highest total society payoff among all three possible coalitions. That is, Eqs. (19) and (20) are valid. Then this coalition is resistant. On the other hand, we assume there exists at least one allocable coalition, and notice that the coalition of Players 1 and 2 is of the highest payoff, hence (18) must be valid. In all, this coalition is both allocable and resistant, therefore stable. □

Remark. (1) From the proof, we can see that the coalition which is stable is the one that maximizes the total payoff of the society.

(2) We assume there exists at least one allocable coalition in this theorem. If none of the coalitions is allocable, it becomes tricky what possible coalition structures would result from the interaction of the players. We demonstrate this point by the following example.

Example 9 (Nonallocable model).

$$\pi_{12} = \pi_{23} = \pi_{13} = \pi_1 = \pi_2 = \pi_3 = 3, \tag{21}$$

$$P_1^N = P_2^N = P_3^N = 1. \tag{22}$$

In this example, forming a coalition with two players is better than working independently, but being left behind is an even better situation. Consider the coalition structure $\{\{1,2\},3\}$. We argue this coalition structure is stable. Note that Player 3 achieves the highest possible payoff and has no incentive to deviate from the status quo. In addition, Players 1 and 2 have no incentive as well. In fact, if Player 1 were to leave the coalition $\{1,2\}$, the best strategy for Players 2 and 3 is to form a coalition and Player 1 will get a payoff 1. However, similarly Player 2 or 3 may also want to deviate from the coalition $\{2,3\}$, which results in endless changes of coalition structures. Thus, we assume that the games analyzed in this paper always admit at least one allocable coalition. This assumption is satisfied for super-additive TU games in coalitional forms.

2.2. *Allocation*

After proving the existence of a stable coalition, now we can discuss the possible allocation between players in a stable coalition.

In the sequel, when discussing allocation, we always indicate a reasonable division of the total payoff in a coalition. The division is reasonable if and only if no participant can be better off after quitting current coalition, hence all participants are willing to stay.

Theorem 10 (Allocation in a stable coalition). *Without loss of generality, assume that the coalition of Players* 1 *and* 2 *is stable.*

(1) *If the coalition is strongly stable, then the allocation should satisfy*

$$\pi_{12}^1 \in [\max(\pi_1, \pi_{13} - \pi_3), \pi_{12} - \max(\pi_2, \pi_{23} - \pi_3)], \tag{23}$$

$$\pi_{12}^2 \in [\max(\pi_2, \pi_{23} - \pi_3), \pi_{12} - \max(\pi_1, \pi_{13} - \pi_3)]. \tag{24}$$

(2) *If the coalition is stable but not strongly stable, then the allocation should satisfy*

$$\pi_{12}^1 \in [\pi_{12} - \max(\pi_2, \pi_{23} - \pi_3), \max(\pi_1, \pi_{13} - \pi_3)], \tag{25}$$

$$\pi_{12}^2 \in [\pi_{12} - \max(\pi_1, \pi_{13} - \pi_3), \max(\pi_2, \pi_{23} - \pi_3)]. \tag{26}$$

Proof. In a stable coalition, Player 1 can earn π_1 if being left behind, and at most $\pi_{13} - \pi_3$ in a coalition with Player 3. In other words, Player 1 can earn no more than $\max(\pi_1, \pi_{13} - \pi_3)$ if he breaks the current situation. Similarly, Player 2 can earn at most $\max(\pi_2, \pi_{23} - \pi_3)$ if he breaks the current situation.

In a strongly stable coalition, if $\pi_{12}^1 < \max(\pi_1, \pi_{13} - \pi_3)$, then

$$\pi_{12}^2 = \pi_{12} - \pi_{12}^1 > \max(\pi_2, \pi_{23} - \pi_3). \tag{27}$$

Player 1 will ask Player 2 for more allocation to enlarge π_{12}^1 to $\max(\pi_1, \pi_{13} - \pi_3)$. Player 2 will agree to this because he can still earn more than $\max(\pi_2, \pi_{23} - \pi_3)$, the most value he can earn if the current coalition breaks. This implies that $\pi_{12}^1 \geq \max(\pi_1, \pi_{13} - \pi_3)$. We can similarly deduce the lower bound of Player 2's allocation and the upper bounds can be deduced from the lower bounds.

In a stable but not strongly stable coalition, if Player 3 intends to form a coalition with Player 1, he can at most provide an allocation of $\pi_{13} - \pi_3$ to Player 1, otherwise he earns less than current situation. So Player 1 can earn at most $\max(\pi_1, \pi_{13} - \pi_3)$ if he quits the current situation. Hence, Player 1's allocation in the coalition with Player 2 should be no more than $\max(\pi_1, \pi_{13} - \pi_3)$. Thus, we deduce that

$$\pi_{12}^2 = \pi_{12} - \pi_{12}^1 \geq \pi_{12} - \max(\pi_1, \pi_{13} - \pi_3). \tag{28}$$

Equation (28) gives the lower bound of Player 2's allocation. The lower bound of Player 1's allocation and upper bound of Player 2's allocation can be similarly derived. □

Remark. (1) In a strongly stable coalition, the allocation is irrelevant to Player 3's action. Any allocation satisfying the condition is a feasible allocation which makes both Players 1 and 2 happy to stay and have totally no interest to cooperate with Player 3.

(2) In a stable but not strongly stable coalition, the allocation is relevant to Player 3's offer to players in the coalition. But there always exists an allocation for Players 1 and 2 to be resistant to Player 3's offer.

(3) $\max(\pi_1, \pi_{13} - \pi_3), \max(\pi_2, \pi_{23} - \pi_3)$ are two important numbers. They are the largest payoff the two players can have if the current coalition breaks. In a strongly stable coalition, these two payoffs can be satisfied simultaneously, and hence they are the lower bound of the allocations. However, in a stable but not strongly stable coalition, at most one of the two payoffs can be satisfied, and hence they become the upper bounds.

(4) In a stable but not strongly stable coalition, notice that the lower bound of Player 1's allocation satisfies

$$\pi_{12} - \max(\pi_2, \pi_{23} - \pi_3) \geq \pi_1, \tag{29}$$

because the coalition is both allocable and resistant. Similarly, we also have

$$\pi_{12} - \max(\pi_1, \pi_{13} - \pi_3) \geq \pi_2. \tag{30}$$

The above two inequalities imply that both players are willing to stay in the coalition since the worst allocation is still larger than their corresponding payoffs when being left behind.

Here we give two examples to illustrate the difference between strongly stable coalition and stable coalition.

Example 11 (Strongly stable coalition).

$$\pi_{12} = \pi_{23} = \pi_{13} = 100, \tag{31}$$

$$\pi_1 = 10, \tag{32}$$

$$\pi_2 = 50, \tag{33}$$

$$\pi_3 = 60. \tag{34}$$

It is easy to check that the coalition of Players 1 and 2 is strongly stable. If Player 1 wants to form a coalition with Player 3, he must offer more than 60 to Player 3, so he himself can earn no more than 40 in such a coalition. Hence $\pi_{12}^1 \geq 40$ is acceptable, since he cannot earn more if working alone or cooperating with Player 3. But $\pi_{12}^1 < 40$ is unacceptable, since he will prefer a coalition with Player 3 in this situation. For Player 2, $\pi_{12}^2 < 50$ is unacceptable, otherwise he may prefer quitting and being left behind. Any feasible allocation satisfying $\pi_{12}^1 \in [40, 50], \pi_{12}^2 \in [50, 60]$ is acceptable by both two players. And any allocation in this set is resistent to any offer by Player 3 because Player 3 can offer at most 40.

Example 12 (Stable but not strongly stable coalition).

$$\pi_{12} = \pi_{23} = \pi_{13} = 100, \tag{35}$$

$$\pi_1 = \pi_2 = 10, \tag{36}$$

$$\pi_3 = 20. \tag{37}$$

It is easy to check that coalition of Players 1 and 2 is stable but not strongly stable. Player 3 can at most provide 80 to Player 1 to distract him away from current coalition with Player 2. If Player 3 negotiates this offer with Player 1, Player 2 will provide an equal amount to Player 1 to maintain the coalition. In this case, Player 2 can earn at least 20. Player 2 is willing to do so, otherwise he may earn only 10 if being left behind. So $\pi_{12}^2 \geq 20$, hence $\pi_{12}^1 \leq 80$. Symmetrically we have $\pi_{12}^1 \geq 20, \pi_{12}^2 \leq 80$. In summary, there will be an allocation satisfying $\pi_{12}^1, \pi_{12}^2 \in [20, 80]$ (depending on Player 3's offer) which makes both players happy to stay.

3. Allocation in a Grand Coalition

3.1. *Allocations*

After studying the stability and allocation of a two-player coalition, we can analyze allocation in a grand coalition. To avoid ambiguity, we may use term "sub-coalition" to indicate a coalition between two players, which is different from a grand coalition.

Denote π_{123} as the optimal total payoff of the grand coalition when all three players make strategies collaboratively. That is,

$$\pi_{123} = \max_{\substack{S_i \in \mathcal{S}_i \\ i=1,2,3}} \sum_{j=1}^{3} P_j(S_1, S_2, S_3). \tag{38}$$

This directly implies the following lemma.

Lemma 13.

$$\pi_{123} \geq P_1^N + P_2^N + P_3^N, \tag{39}$$

$$\pi_{123} \geq \pi_{12} + \pi_3, \tag{40}$$

$$\pi_{123} \geq \pi_{13} + \pi_2, \tag{41}$$

$$\pi_{123} \geq \pi_{23} + \pi_1. \tag{42}$$

Define $\pi_{123}^1, \pi_{123}^2, \pi_{123}^3$ *as the allocation to Players* 1, 2, *and* 3, *respectively. Hence,*

$$\pi_{123} = \pi_{123}^1 + \pi_{123}^2 + \pi_{123}^3. \tag{43}$$

Now we analyze what conditions these allocations should satisfy.

If there is at least one allocable coalition, we have proved in last section that there definitely exists at least one stable coalition. To make things simple, we first study the case that there exists exactly one stable coalition. Without loss of generality, we assume the stable coalition is between Players 1 and 2.

Theorem 14 (Allocation in grand coalition). *In a grand coalition with only one stable coalition between Players* 1 *and* 2, *if the coalition is strongly stable, the allocation should satisfy*

$$\pi_{123}^1 \in [\max(\pi_1, \pi_{13} - \pi_3), \pi_{123} - \pi_3 - \max(\pi_2, \pi_{23} - \pi_3)], \tag{44}$$

$$\pi_{123}^2 \in [\max(\pi_2, \pi_{23} - \pi_3), \pi_{123} - \pi_3 - \max(\pi_1, \pi_{13} - \pi_3)], \tag{45}$$

$$\pi_{123}^1 + \pi_{123}^2 \in [\pi_{12}, \pi_{123} - \pi_3], \tag{46}$$

$$\pi_{123}^3 \in [\pi_3, \pi_{123} - \pi_{12}]. \tag{47}$$

Otherwise, the allocation should satisfy

$$\pi_{123}^1 \in [\pi_{12} - \max(\pi_2, \pi_{23} - \pi_3), \pi_{123} - \pi_{12} - \pi_3 + \max(\pi_1, \pi_{13} - \pi_3)], \tag{48}$$

$$\pi_{123}^2 \in [\pi_{12} - \max(\pi_1, \pi_{13} - \pi_3), \pi_{123} - \pi_{12} - \pi_3 + \max(\pi_2, \pi_{23} - \pi_3)], \tag{49}$$

$$\pi_{123}^1 + \pi_{123}^2 \in [\pi_{12}, \pi_{123} - \pi_3], \tag{50}$$

$$\pi_{123}^3 \in [\pi_3, \pi_{123} - \pi_{12}]. \tag{51}$$

Proof. If the grand coalition breaks, Players 1 and 2 will form a stable coalition and Player 3 will be left behind. Each player's allocation should be no less than his minimal payoff if the grand coalition breaks, otherwise he has incentive to quit. On the other hand, a player whose allocation is no less than his minimal payoff outside the grand coalition has no incentive to leave. Hence, Theorem 10 specifies allocations' lower bounds. Moreover, Players 1 and 2's total payoff should be larger than π_{12}. Otherwise, they can deviate from the grand allocation to form a coalition such that their payoffs can be increased simultaneously.

The upper bounds are deduced from the lower bounds. □

Remark. Lemma 13 implies that

$$\pi_{123}^1 = \pi_{123} - \max(\pi_2, \pi_{23} - \pi_3) - \pi_3, \tag{52}$$

$$\pi_{123}^2 = \max(\pi_2, \pi_{23} - \pi_3), \tag{53}$$

$$\pi_{123}^3 = \pi_3 \tag{54}$$

always satisfies (44)–(47) and (48)–(51), which means that the four constraints are compatible, and the feasible allocation set is nonempty.

Similarly, we analyze the allocation for cases in which there are more than one stable coalitions.

Theorem 15 (Allocation in grand coalition). *In a grand coalition,*

(1) *if there exist two stable coalitions, say the coalition of Players* 1 *and* 2 *and the coalition of Players* 1 *and* 3, *then the allocation should satisfy*

$$\pi_{123}^1 \in [\pi_{13} - \max(\pi_3, \pi_{23} - \pi_2), \pi_{123} - \pi_2 - \pi_3], \tag{55}$$

$$\pi_{123}^2 \in [\pi_2, \pi_{123} - \pi_{13}], \tag{56}$$

$$\pi_{123}^3 \in [\pi_3, \pi_{123} - \pi_{12}], \tag{57}$$

$$\pi_{123}^1 + \pi_{123}^2 \in [\pi_{12}, \pi_{123} - \pi_3], \tag{58}$$

$$\pi_{123}^1 + \pi_{123}^3 \in [\pi_{13}, \pi_{123} - \pi_2], \tag{59}$$

(2) *if all three coalitions are stable, then the allocation should satisfy*

$$\pi_{123}^i \in [\pi_i, \pi_{123} - \pi_{jk}], \tag{60}$$

$$\pi_{123}^i + \pi_{123}^j \in [\pi_{ij}, \pi_{123} - \pi_k], \tag{61}$$

where $\{i, j, k\} = \{1, 2, 3\}$.

Proof. (1) In Case 1, there exists two stable coalitions of Players 1 and 2 and Players 1 and 3. The proof of Theorem 8 implies that

$$\pi_{12} + \pi_3 = \pi_{13} + \pi_2, \tag{62}$$

which implies (17) is satisfied if and only if

$$\pi_{13} \geq (\pi_{12} - \pi_2) + (\pi_{23} - \pi_2). \tag{63}$$

Following Theorem 7, the coalition between Players 1 and 2 is strongly stable if and only if the coalition between Players 1 and 3 is strongly stable.

- If both of the coalitions are not strongly stable, then in the coalition between Players 1 and 2, Player 1's minimal payoff is $\pi_{12} - \max(\pi_2, \pi_{23} - \pi_3)$, while in the coalition with Player 3, his minimal payoff is $\pi_{13} - \max(\pi_3, \pi_{23} - \pi_2)$. Notice these two values are equal because of (62).

  For Player 2, his minimal payoff in a coalition with Player 1 is

$$\pi_{12} - \max(\pi_1, \pi_{13} - \pi_3) = \pi_{12} - \pi_{13} + \pi_3 = \pi_2. \tag{64}$$

  The first equality follows from allocability of the coalition between Players 1 and 3, and the second equality follows from (62). Similar analysis applies to Player 3.
- If both of the coalitions are strongly stable,

$$\begin{aligned} \pi_{12} - \max(\pi_2, \pi_{23} - \pi_3) \leq \pi_{12} - \pi_2 &= \pi_{13} - \pi_3 \\ &= \max(\pi_1, \pi_{13} - \pi_3), \end{aligned} \tag{65}$$

  where the first equality is due to (62) and the second one is due to the allocability of the coalition between Players 1 and 3. Hence, Eq. (13) implies that $\pi_{12} - \max(\pi_2, \pi_{23} - \pi_3) = \max(\pi_1, \pi_{13} - \pi_3)$. Equation (65) implies that Player 1's allocation in the coalition with Player 2 is fixed at $\pi_{12} - \max(\pi_2, \pi_{23} - \pi_3)$, which is equal to $\pi_{12} - \pi_2$. Therefore, Player 2's allocation is π_2. Similar analysis applies to the coalition between Players 1 and 3.

Moreover, the total allocation for Players 1 and 2 or Players 1 and 3 should be no less than the total payoff of corresponding coalition if the grand coalition breaks.

Upper bounds are deduced from lower bounds.

(2) In Case 2, all sub-coalitions are stable. The proof of Theorem 8 implies that

$$\pi_{12} + \pi_3 = \pi_{13} + \pi_2 = \pi_{23} + \pi_1. \tag{66}$$

By a similar argument as in Case 1, Player 1's minimal payoff in a coalition with Player 2 is

$$\pi_{12} - \max(\pi_2, \pi_{23} - \pi_3) = \pi_{12} - \pi_{23} + \pi_3 = \pi_1. \tag{67}$$

The first equality is due to the allocability of the coalition between Players 2 and 3, and the second equality follows from (66). Similar situation happens in a coalition with Player 3.

In summary, Player 1's minimal payoff is π_1 if the grand coalition breaks. Similarly, we can deduce Players 2 and 3's minimal payoffs.

Moreover, total allocation of any two players in the grand coalition should be no less than total payoff of corresponding coalition if the grand coalition breaks.

Upper bounds are deduced from lower bounds. □

Remark. (1) In Case 1, notice that allocation $(\pi_{123}^1, \pi_{123}^2, \pi_{123}^3) = (\pi_{123} - \pi_2 - \pi_3, \pi_2, \pi_3)$ is always a feasible allocation.

(2) However, in Case 2, the bounds may not be compatible. For instance, if

$$\pi_1 = \pi_2 = \pi_3 = 1, \tag{68}$$

$$\pi_{12} = \pi_{13} = \pi_{23} = 3, \tag{69}$$

$$\pi_{123} = 4, \tag{70}$$

a feasible allocation cannot be found. This implies that the grand coalition is not stable.

3.2. *Comparison with the core*

Now we can compare our solution with a classical solution concept: the core. The core is a solution set defined by (1). Specifically, in our model, the characteristic function v is defined as

$$v(\emptyset) = 0, \tag{71}$$

$$v(\{i\}) = \pi_i, \quad i = 1, 2, 3, \tag{72}$$

$$v(\{i, j\}) = \pi_{ij}, \quad i, j \in \{1, 2, 3\}, \ i < j, \tag{73}$$

$$v(\{1, 2, 3\}) = \pi_{123}. \tag{74}$$

Consequently, an allocation $(\pi_{123}^1, \pi_{123}^2, \pi_{123}^3)$ is in core C if and only if

$$\sum_{i=1}^{3} \pi_{123}^i = \pi_{123}, \tag{75}$$

$$\pi_{123}^i \geq \pi_i, \quad i = 1, 2, 3, \tag{76}$$

$$\pi_{123}^i + \pi_{123}^j \geq \pi_{ij}, \quad i, j \in \{1, 2, 3\}, \ i < j. \tag{77}$$

Theorem 16. *If there exists only one stable coalition between Players* 1 *and* 2 *and it is strongly stable, then an allocation satisfying* (44)–(47) *belongs to* C.

Equations (75)–(77) can be deduced directly from (43) and (44)–(47). The detailed proof is omitted here. We give the following example to show that the opposite is invalid, which means there may exist some allocation contained in the core, but not satisfying our definition. Hence our solution is not equivalent to the classical core solution.

Example 17.

$$\pi_{123} = 7, \tag{78}$$

$$\pi_{23} = \pi_{13} = 3, \quad \pi_{12} = 5, \tag{79}$$

$$\pi_1 = \pi_2 = \pi_3 = 1, \tag{80}$$

The setting is the same as the first example given in the introduction. As we illustrate there, the allocation $(\pi_{123}^1, \pi_{123}^2, \pi_{123}^3) = (1.2, 3.9, 1.9)$ belongs to the core. However, it does not satisfy (44)–(47).

Theorem 18. *If there exists only one stable sub-coalition between Players* 1 *and* 2 *and it is not strongly stable, the core* C *and the set* (48)–(51) *do not imply each other.*

We give two examples to verify Theorem 18:

Example 19.

$$\pi_{123} = 8, \tag{81}$$

$$\pi_1 = \pi_2 = \pi_3 = 1, \tag{82}$$

$$\pi_{13} = \pi_{23} = 5, \quad \pi_{12} = 6, \tag{83}$$

The setting is the same as the second example given in the introduction. $(\pi_{123}^1, \pi_{123}^2, \pi_{123}^3) = (3, 3.5, 1.5)$ satisfies (48)–(51) but does not belong to the core. We can check that the coalition between Players 1 and 3 is not stable but the coalition between Players 1 and 2 is. Hence, Player 3 has no incentive to deviate from the grand coalition with Player 1 at the first place.

Example 20.

$$\pi_{123} = 51, \tag{84}$$

$$\pi_1 = \pi_2 = \pi_3 = 10, \tag{85}$$

$$\pi_{13} = \pi_{23} = 30, \quad \pi_{12} = 31, \tag{86}$$

Notice $(\pi_{123}^1, \pi_{123}^2, \pi_{123}^3) = (10, 21, 20)$ is an allocation in the core. But it does not satisfy (48)–(51) since Player 1 can still threat to quit the grand coalition because he knows after this he can have a payoff of at least 11 in a stable coalition with Player 2.

Theorem 21. *If two of the three coalitions are stable, the core* C *and the set* (55)–(59) *do not imply each other.*

We give an example to illustrate the difference:

Example 22.

$$\pi_{123} = 7, \tag{87}$$

$$\pi_1 = \pi_2 = \pi_3 = 1, \tag{88}$$

$$\pi_{12} = \pi_{13} = 4, \quad \pi_{23} = 3, \tag{89}$$

$(\pi_{123}^1, \pi_{123}^2, \pi_{123}^3) = (1, 3, 3)$ is an allocation in the core. However, it does not satisfy (55)–(59) because Player 1 wants to deviate from the grand coalition to get a payoff of at least 2.

$(\pi_{123}^1, \pi_{123}^2, \pi_{123}^3) = (4.8, 1.1, 1.1)$ is not included in the core because the total payoff of Players 2 and 3 is less than π_{23}. But they know their coalition is not stable. Hence, $(\pi_{123}^1, \pi_{123}^2, \pi_{123}^3) = (4.8, 1.1, 1.1)$ is an allocation that satisfies (55)–(59).

Theorem 23. *If all three coalitions are stable, the core C and the set* (60) *and* (61) *are equivalent.*

4. Application

In previous sections, all the results are based on parameters $\pi_1, \pi_2, \pi_3, \pi_{12}, \pi_{23}$, π_{13}, π_{123}. We did not specify what payoff functions are used to obtain these values. In this section, we study a pricing game where every player sells a similar product and they compete on prices. Each player's price not only affects his own product's demand but also affects the products' demand of the other players.

The strategy S_i is the price set to the product. The demand of player i is $l_i - q_{ii}S_i + \sum_{j \neq i} q_{ij}S_j$ and the payoff functions are

$$P_i(S_1, S_2, S_3) = \left(l_i - q_{ii}S_i + \sum_{j \neq i} q_{ij}S_j \right) S_i, \quad i = 1, 2, 3, \tag{90}$$

where l, q are positive parameters and $\begin{bmatrix} q_{11} & -q_{12} & -q_{13} \\ -q_{21} & q_{22} & -q_{23} \\ -q_{31} & -q_{32} & q_{33} \end{bmatrix}$ is an M-matrix. The M-matrix assumption is justified in Lu and Simchi-Levi (2009). A nonsingular square matrix is called M-matrix if all off-diagonal entries are less than or equal to zero and its inverse is non-negative.

To make things simple, we analyze a simpler model with symmetric quadratic terms here.

$$P_i(S_1, S_2, S_3) = \left(l_i - dS_i + \sum_{j \neq i} qS_j \right) S_i, \quad i = 1, 2, 3. \tag{91}$$

A player differs other players only on linear coefficient l_i. Obviously, a player with a larger l_i is stronger. Moreover, matrix $\begin{bmatrix} d & -q & -q \\ -q & d & -q \\ -q & -q & d \end{bmatrix}$ is an M-matrix, which is equivalent to $d > 2q > 0$.

4.1. *Nash equilibrium in coalition compared with no-coalition*

When there exists no coalition, by solving equations

$$\frac{\partial P_i}{\partial S_i} = 0, \quad i = 1, 2, 3, \tag{92}$$

we get

$$S_1^N = \frac{(2d-q)l_1 + ql_2 + ql_3}{4d^2 - 4dq - 2q^2} > 0, \tag{93}$$

$$S_2^N = \frac{(2d-q)l_2 + ql_1 + ql_3}{4d^2 - 4dq - 2q^2} > 0, \tag{94}$$

$$S_3^N = \frac{(2d-q)l_3 + ql_1 + ql_2}{4d^2 - 4dq - 2q^2} > 0, \tag{95}$$

which is the Nash equilibrium. At this point, payoff for each player is

$$P_1^N = \frac{d((2d-q)l_1 + ql_2 + ql_3)^2}{(4d^2 - 4dq - 2q^2)^2}, \tag{96}$$

$$P_2^N = \frac{d((2d-q)l_2 + ql_1 + ql_3)^2}{(4d^2 - 4dq - 2q^2)^2}, \tag{97}$$

$$P_3^N = \frac{d((2d-q)l_3 + ql_1 + ql_2)^2}{(4d^2 - 4dq - 2q^2)^2}. \tag{98}$$

When there is a coalition between Players 1 and 2, we solve

$$\frac{\partial(P_1 + P_2)}{\partial S_1} = \frac{\partial(P_1 + P_2)}{\partial S_2} = \frac{\partial P_3}{\partial S_3} = 0 \tag{99}$$

and get

$$S_1^* = \frac{4d^2 l_1 + 2dq(2l_2 + l_3) + q^2(l_2 + 2l_3 - l_1)}{4(2d^3 - 3dq^2 - q^3)} > 0, \tag{100}$$

$$S_2^* = \frac{4d^2 l_2 + 2dq(2l_1 + l_3) + q^2(l_1 + 2l_3 - l_2)}{4(2d^3 - 3dq^2 - q^3)} > 0, \tag{101}$$

$$S_3^* = \frac{2(d-q)l_3 + ql_1 + ql_2}{4d^2 - 4dq - 2q^2} > 0, \tag{102}$$

which is the new Nash equilibrium.

We compare the total payoff of Players 1 and 2 with $P_1^N + P_2^N$

$$
\begin{aligned}
&P_1(S_1^*, S_2^*, S_3^*) + P_2(S_1^*, S_2^*, S_3^*) - (P_1^N + P_2^N) \\
&\quad = q^2(8d^5(d-q)(l_1^2 + l_2^2) + 16d^3q(3d^2 - 3dq - q^2)l_1l_2 \\
&\qquad + d^2q^2(16d^2 - 16dq - 11q^2)(l_1^2 + l_2^2) \\
&\qquad + 4q^2(d^4 + 2d^3q - d^2q^2 - 3dq^3 - q^4)l_3^2 \\
&\qquad + 8dq(d^4 + 2d^3q - d^2q^2 - 3dq^3 - q^4)l_3(l_1 + l_2) \\
&\qquad + 6d^2q^4l_1l_2 + q^5(2d+q)(l_1 - l_2)^2/ \\
&\qquad (8(d+q)(4d^4 - 6d^3q - 2d^2q^2 + 3dq^3 + q^4)^2) > 0. \qquad (103)
\end{aligned}
$$

Notice that each term in the parentheses of the numerator is positive except for the last one, which is still non-negative. This implies that coalition provides a strictly larger total payoff for Players 1 and 2

$$\pi_{12} > P_1^N + P_2^N. \qquad (104)$$

We can also compare Player 3's current payoff with P_3^N

$$
\begin{aligned}
P_3(S_1^*, S_2^*, S_3^*) - P_3^N &= dq^2(2(l_1 + l_2)l_3(4d^4 - 6d^3q + 2d^2q^2 - q^4) \\
&\quad + (l_1 + l_2)^2dq(4d^2 - 3dq - 2q^2) \\
&\quad + l_3^2q(8d^3 - 12d^2q - 3q^3))/ \\
&\quad (4(4d^4 - 6d^3q - 2d^2q^2 + 3dq^3 + q^4)^2) > 0. \qquad (105)
\end{aligned}
$$

Notice that each term in the parentheses of the numerator is positive. This implies Player 3 also benefits from the coalition of Players 1 and 2

$$\pi_3 > P_3^N. \qquad (106)$$

Similarly, we have the same results on the other two coalitions.

Observation 24. All three players can benefit from a coalition between any two players.

4.2. *Allocability*

We have

$$
\begin{aligned}
\pi_{13} - \pi_1 - \pi_3 &= q^2(d^2(l_1^2 + l_3^2 - 4l_1l_3 - 2l_2(l_1 + l_3)) \\
&\quad - 2dq(3l_1^2 + 3l_3^2 - 4l_1l_3 + 2l_2(l_1 + l_2 + l_3)) \\
&\quad + q^2(l_1^2 + l_3^2 - 2l_1l_3 - 4l_2^2))/(8(d+q)(2d^2 - 2dq - q^2)^2) \qquad (107)
\end{aligned}
$$

and symmetric form of $\pi_{12} - \pi_1 - \pi_2$ and $\pi_{23} - \pi_2 - \pi_3$.

If $l_1 = l_2 \geq l_3$, then

$$\pi_{13} - \pi_1 - \pi_3 = \pi_{23} - \pi_2 - \pi_3$$
$$= q^2(d^2(l_3^2 - l_1^2 - 6l_1l_3) - 2dq(7l_1^2 + 3l_3^2 - 2l_1l_3)$$
$$+ q^2(l_3^2 - 2l_1l_3 - 3l_1^2))/(8(d+q)(2d^2 - 2dq - q^2)^2) < 0 \quad (108)$$

and

$$\pi_{12} - \pi_1 - \pi_2 = q^2(d^2(-2l_1^2 - 4l_1l_3) - 4dq(l_1 + l_3)^2 - 4q^2l_3^2)/$$
$$(8(d+q)(2d^2 - 2dq - q^2)^2) < 0, \quad (109)$$

then all three coalitions are nonallocable.

Observation 25. If the two strongest players have equal strength, then all coalitions are nonallocable.

Cases with at least one allocable coalition are also possible. An example is

$$l_1 = 10, \quad l_2 = 1, \quad l_3 = 1, \quad d = 10, \quad q = 1, \quad (110)$$

which makes $\pi_{13} - \pi_1 - \pi_3$ positive. Hence, the coalition between Players 1 and 3 is allocable.

Observation 26. Nonallocable and allocable cases both exist.

4.3. *Stable coalition*

As shown in previous subsection that $l_1 = l_2 > l_3 > 0$ leads to nonallocable cases. Hence, we assume that $l_1 > l_2 \geq l_3 > 0$ in this subsection.

Notice that

$$\pi_{13} + \pi_2 - \pi_{23} - \pi_1 = q^2(l_1 - l_2)(2dq(d-q)(l_1 + l_2 - 2l_3) + 4dql_3$$
$$+ q^2(5l_1 + 5l_2 - 2l_3))/(8(d+q)(2d^2 - 2dq - q^2)^2) > 0. \quad (111)$$

We have the following observation:

Observation 27. If $l_1 > l_2 \geq l_3 > 0$, the coalition between the two weakest players (2 and 3) is not resistent, hence it is nonstable.

However, the other two coalitions can be stable. It is easy to verify that when $(l_1, l_2, l_3, d, q) = (8, 1, 1, 14, 1)$, the coalition between Players 1 and 3 is stable, while when $(l_1, l_2, l_3, d, q) = (8, 1, 1, 15, 1)$, coalition between Players 1 and 2 is stable.

Observation 28. If $l_1 > l_2 \geq l_3 > 0$, stable coalition can exist between Players 1 and 2 or Players 1 and 3.

4.4. *Allocation in a coalition structure*

For Player 1, we have

$$\pi_1 = \frac{d(2(d-q)l_1 + ql_2 + ql_3)^2}{4(2d^2 - 2dq - q^2)^2}, \tag{112}$$

$$\begin{aligned}\pi_{13} - \pi_3 = {} & ((8d^4 - 8d^3q - 6d^2q^2 + 2dq^3 + q^4)l_1^2 \\ & + (2d^2q^2 - 2dq^3 - 4q^4)l_2^2 \\ & + (4d^2q^2 - 4dq^3 + q^4)l_3^2 + (8d^3q - 4d^2q^2 - 12dq^3)l_1l_2 \\ & + (8d^3q - 8d^2q^2 - 2q^4)l_1l_3)/(8(d+q)(2d^2 - 2dq - q^2)^2), \end{aligned} \tag{113}$$

$$\begin{aligned}\pi_{12} - \pi_2 = {} & ((4d^2q^2 - 4dq^3 + q^4)l_1^2 + (8d^4 - 8d^3q - 6d^2q^2 + 2dq^3 + q^4)l_2^2 \\ & + (2d^2q^2 - 2dq^3 - 4q^4)l_3^2 + (8d^3q - 8d^2q^2 - 2q^4)l_1l_2 \\ & + (8d^3q - 4d^2q^2 - 12dq^3)l_2l_3)/(8(d+q)(2d^2 - 2dq - q^2)^2), \end{aligned} \tag{114}$$

$$\begin{aligned}\pi_{12} - (\pi_{23} - \pi_3) = {} & ((8d^4 - 8d^3q - 6d^2q^2 + 6dq^3 + 5q^4)l_1^2 \\ & + (2d^2q^2 + 2dq^3)l_2^2 + (4dq^3 - 5q^4)l_3^2 \\ & + (8d^3q - 4d^2q^2 + 4dq^3 - 2q^4)l_1l_2 + (8d^3q - 8dq^3)l_1l_3 \\ & + (8d^2q^2 - 8dq^3 + 2q^4)l_2l_3)/(8(d+q)(2d^2 - 2dq - q^2)^2). \end{aligned} \tag{115}$$

Notice all coefficients are positive, which implies that all above four values have positive partial derivatives with respect to l_1, l_2, l_3. Hence, $\pi_1, \max(\pi_1, \pi_{13} - \pi_3)$ and $\pi_{12} - \max(\pi_2, \pi_{23} - \pi_2)$ increase with l_1,l_2, and l_3. Symmetric analysis also applies to Players 2 and 3. Theorem 10 implies the following observation.

Observation 29. When one player becomes stronger (his parameter l_i is larger), every player's payoff (both lower bounds and upper bounds) increases.

4.5. *Grand coalition*

In a grand coalition, solving

$$\frac{\partial(P_1 + P_2 + P_3)}{\partial S_i} = 0, \quad i = 1, 2, 3, \tag{116}$$

gives

$$S_i^G = \frac{(d+q)l_i + q\sum_{j\neq i} l_j}{2(d-2q)(d+q)} > 0, \quad i = 1, 2, 3. \tag{117}$$

Notice that

$$\sum_{i=1}^{3} P_i(S_1^G, S_2^G, S_3^G) - (\pi_{12} + \pi_3) = q^2(d(2d^2 - 2dq + 5q^2)(l_1^2 + l_2^2)$$
$$+ 2(2d^3 - 4d^2q + 7dq^2 - 5q^3)l_3^2 + 2d(2d^2 - 2dq + 5q^2)l_1l_2$$
$$+ 4q(2d^2 - 4dq + q^2)l_3(l_1 + l_2))/(8(d^2 - dq - 2q^2)(2d^2 - 2dq - q^2)^2) > 0. \tag{118}$$

Similar analysis applies to the comparison with $\pi_{13} + \pi_2$ and $\pi_{23} + \pi_1$. This implies that total payoff of the grand coalition is larger than the payoff of any other coalition structures. Hence,

$$\pi_{123} = \sum_{i=1}^{3} P_i(S_1^G, S_2^G, S_3^G), \tag{119}$$

To analyze the payoff bounds in a grand coalition, we first present a lemma. Notice that the role of three players are completely symmetric

Lemma 30. *The following values and their symmetric counterparts have positive partial derivatives with respect to* l_1, l_2, l_3, *respectively:* $\pi_1, \pi_{12}, \pi_{13} - \pi_3, \pi_{12} - (\pi_{23} - \pi_3), \pi_{123} - \pi_1, \pi_{123} - \pi_{12}, \pi_{123} - \pi_1 - \pi_2, \pi_{123} - \pi_{12} - \pi_3 + \pi_1, \pi_{123} - \pi_{12} - \pi_3 + (\pi_{13} - \pi_3)$.

Proof. Formulas of $\pi_1, \pi_{12}, \pi_{13} - \pi_3, \pi_{12} - (\pi_{23} - \pi_3)$ are given in previous subsections, which imply that they satisfy the condition. Equation (118) shows that $\pi_{123} - \pi_{12} - \pi_3$ also satisfies this condition. Hence $(\pi_{123} - \pi_{23} - \pi_1) + \pi_{23}, (\pi_{123} - \pi_{12} - \pi_3) + \pi_3, (\pi_{123} - \pi_{13} - \pi_2) + (\pi_{13} - \pi_1), (\pi_{123} - \pi_{12} - \pi_3) + \pi_1$ and $(\pi_{123} - \pi_{12} - \pi_3) + (\pi_{13} - \pi_3)$ satisfy the condition. □

This lemma and Theorems 14 and 15 imply that

Theorem 31. *In a grand coalition, every player's payoff (both lower bounds and upper bounds) increases if one player becomes stronger (his parameter* l_i *is larger).*

5. Conclusion

In this paper, we introduced a stability concept in a three-player game and showed the existence of a stable coalition. Based on this result, we studied allocations in two-player and three-player coalitions. We discussed how our solution differentiates with the core and applied it on a stylized price competition model to get some interesting observations.

For a game with more than three players, the analysis will be much more complicated as the number of coalitions exponentially increases. As a potential future research, we need a smart way to deal with it. Nevertheless, we believe that the stability concept introduced in paper can be generalized and offer a new aspect to analyze cooperative games.

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