A Counter-Example to a Conjecture of Ben-Tal, Nemirovski and Roos

Ya-xiang Yuan

Journal of the Operations Research Society of China

ISSN 2194-668X
Volume 1
Number 1

J. Oper. Res. Soc. China (2013) 1:155-157
DOI 10.1007/s40305-012-0001-8

Springer

JORC (2013) 1:155–157
DOI 10.1007/s40305-012-0001-8

A Counter-Example to a Conjecture of Ben-Tal, Nemirovski and Roos

Ya-xiang Yuan

Received: 27 September 2012 / Revised: 10 November 2012 / Accepted: 22 November 2012 /
Published online: 25 January 2013

Abstract In this short note, we present a counter-example to a conjecture made by Ben-Tal et al. in SIAM J. Optim. 13: 535–560, 2002.

Keywords Counter-example · Conjecture · Quadratic constrained optimization · Robust solution

1 Introduction

About 10 years ago, Ben-Tal, Nemirovski and Roos [1] made the following conjecture:

Conjecture 1.1 (Ben-Tal, Nemirovski and Roos, 2002) Let $B \in \Re^{n\times n}$ be a symmetric matrix, and let $\xi = (\xi_1, \cdots, \xi_n) \in \Re^n$ with the coordinates ξ_i of ξ being independently identically distributed random variables with

$$\Pr(\xi_i = 1) = \Pr(\xi_i = -1) = \frac{1}{2}. \tag{1.1}$$

Then,

$$\Pr\bigl(\xi^T B\xi \leqslant \mathrm{Tr}(B)\bigr) \geqslant \frac{1}{4}. \tag{1.2}$$

This work was partially supported by Chinese NSF (Nos. 10831006, 11021101) and by CAS (No. kjcx-yw-s7).

Y.-x. Yuan (✉)
State Key Laboratory of Scientific/Engineering Computing, Institute of Computational Mathematics and Scientific/Engineering Computing, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Zhong Guan Cun Donglu 55, Beijing 100190, P.R. China
e-mail: yyx@lsec.cc.ac.cn

The estimation of a quadratic form plays an important role in the study of robust optimization. See Ben-Tal, Nemirovski and Roos [1] and the reference therein.

Therefore, a positive answer to the above conjecture would be very useful in analyzing robust solutions of uncertain quadratic and conic-quadratic optimization problems. Define the lower bound

$$y(n) = \inf_{B \in \mathcal{S}^{n\times n}} \Pr\big(\xi^{\mathrm{T}} B \xi \leqslant \mathrm{Tr}(B)\big), \tag{1.3}$$

where $\mathcal{S}^{n\times n} = \{B \in \Re^{n\times n}, B = B^{\mathrm{T}}\}$. The conjecture says that $y(n) \geqslant \frac{1}{4}$ for all n. In fact, there are a few results proving lower bounds for $y(n)$. In [1], it was proved that $y(n) \geqslant \frac{1}{8n^2}$. Lower bounds of $\frac{1}{2n}$ and $\frac{3}{100}$ were proved in [2] and [3], respectively.

Unfortunately, by constructing a simple example, we give a negative answer to Conjecture 1.1. We present our counter-example in the next section.

2 The Counter Example

Our example is an example where $n = 6$. We denote the vector $e = (1, 1, 1, 1, 1, 1)^{\mathrm{T}} \in \Re^6$ and let

$$B = -ee^{\mathrm{T}} = \begin{pmatrix} -1 & -1 & -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 & -1 & -1 \end{pmatrix}. \tag{2.1}$$

For the above B, we can easily calculate

$$\Pr\big(\xi^{\mathrm{T}} B \xi \leqslant \mathrm{Tr}(B)\big) = \frac{14}{64} = \frac{7}{32} < \frac{1}{4}. \tag{2.2}$$

Therefore, we can see that $y(6) \leqslant \frac{14}{64} < \frac{1}{4}$, which is a counter-example to Conjecture 1.1. Direct computations indicate that it is very likely to have

$$\min_{n\in\{1,2,\cdots\}} \big\{\Pr\big(\xi^{\mathrm{T}} B \xi \leqslant \mathrm{Tr}(B)\big) \bigm| B = -ee^{\mathrm{T}} \in \Re^{n\times n}\big\} = \frac{14}{64}.$$

Thus, it would be interesting to know whether $\frac{14}{64}$ is a lower bound for

$$y^{(1)}(n) = \inf_{B\in\mathcal{S}^{n\times n}, \mathrm{Rank}(B)=1} \Pr\big(\xi^{\mathrm{T}} B \xi \leqslant \mathrm{Tr}(B)\big).$$

3 Discussions

In [1], another conjecture was also made:

Conjecture 3.1 (Ben-Tal, Nemirovski and Roos, 2002) Let $x = (x_1, \cdots, x_n)$ and $\xi = (\xi_1, \cdots, \xi_n) \in \Re^n$. If $||x||_2 = 1$ and the coordinates ξ_i of ξ being independently identically distributed random variables with (1.1). Then one has,

$$\Pr\big(\big|\xi^{\mathrm{T}} x\big| \leqslant 1\big) \geqslant \frac{1}{2}. \tag{3.1}$$

The above conjecture is closely related to Conjecture 1.1. Indeed, it is easy to see that $\Pr(|\xi^{\mathrm{T}}x| \leqslant 1)$ is a special value of $\Pr(\xi^{\mathrm{T}}B\xi \leqslant \mathrm{Tr}(B))$ when $B = \alpha xx^{\mathrm{T}}$ for any $\alpha > 0$. Thus, Conjecture 3.1 is for the special case when B are restricted to rank-one symmetric positive semi-definite matrices, while Conjecture 1.1 is for general symmetric matrices B. Our example uses a rank-one negative semi-definite matrix. Thus, it is not a counter-example to Conjecture 3.1. Though (3.1) was proved in [1] if the righthand side term $\frac{1}{2}$ is replaced by $\frac{1}{3}$, Conjecture 3.1 remains an interesting open problem.

Acknowledgements I would like to thank Dr Zhening Li for drawing my attentions to these interesting conjectures.

References

[1] Ben-Tal, A., Nemirovski, A., Roos, C.: Robust solutions of uncertain quadratic and conic-quadratic problems. SIAM J. Optim. **13**, 535–560 (2002)

[2] Derinkuyu, K., Pinar, M., Camci, A.: An improved probability bound for the approximate S-lemma. Oper. Res. Lett. **35**, 743–746 (2007)

[3] He, S., Luo, Z.-Q., Nie, J., Zhang, S.: Semidefinite relaxation bounds for indefinite homogeneous quadratic optimization. SIAM J. Optim. **19**, 503–523 (2008)