A new linearization method for quadratic assignment problems*

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Abstract

The quadratic assignment problem (QAP) is one of the great challenges in combinatorial optimization. Linearization for QAP is to transform the quadratic objective function into a linear one. Numerous QAP linearizations have been proposed, most of which yield mixed integer linear programs (MILP). Kauffmann and Broeckx's linearization (KBL) is the current smallest one in terms of the number of variables and constraints. In this paper, we give a new linearization, which has the same size as KBL. Our linearization is more efficient in terms of the tightness of the continuous relaxation. Furthermore, the continuous relaxation of our linearization leads to an improvement to the Gilmore-Lawler bound (GLB). We also give a corresponding cutting plane heuristic method for QAP and demonstrate its superiority by numerical results.

Keywords: quadratic assignment problem, linearization, mixed integer linear program, cutting plane

1 Introduction

The quadratic assignment problem (QAP) is one of the great challenges in combinatorial optimization. For comprehensive surveys of QAPs, we refer to [7, 11, 19]. A nice review on recent advances is given by [3]. The following formulation of QAP was used initially by Koopmans and Beckmann[16].

$$\mathbf{QAP}: \quad \min \quad f(X) = \sum_{i,j,k,l} a_{ik} b_{jl} x_{ij} x_{kl} \tag{1.1}$$

$$s.t. \quad \sum_{j} x_{ij} = 1, \quad i = 1, ..., n, \tag{1.2}$$

$$\sum_{i} x_{ij} = 1, \quad j = 1, ..., n, \tag{1.3}$$

$$x_{ij} \in \{0, 1\}, \quad i, j = 1, ..., n, \tag{1.4}$$

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where $A = (a_{ik})_{n\times n}$ corresponds to the flow matrix and $B = (b_{jl})_{n\times n}$ corresponds to the distance matrix in the facility location application. $x_{ij} = 1$ means facility i being placed in location j. For simplicity we sometimes use the notation $X = (x_{ij})_{n\times n} \in \Pi_n$ to represent (1.2), (1.3), (1.4), where Π_n denotes the set of $n \times n$ permutation matrices. In this paper, the optimal value of the QAP problem (1.1)-(1.4) is denoted by QAP(A,B).

It is well known that QAP is NP-hard [20]. In that paper it was also proved that finding an $\epsilon-$approximate solution of QAP is also NP-hard. Many well-known NP-hard problems such as travelling salesman problem, the graph partitioning problem, the maximum clique problem, graph isomorphism and largest common subgraph problem can be reformulated as special QAPs. The practice also shows that QAP is extremely difficult to solve to optimality. Problems of size $n \geq 20$ are currently considered huge problems.

Linearization is the first attempt to solve QAP and is achieved by introducing new variables and new linear (and binary) constraints. Then existing methods for (mixed) linear integer programming (MILP) can be applied. MILP formulations also provide linear programming (LP) relaxations which give lower bounds. In this paper, we will give a new linearization, and show that it has the same size as Kauffmann and Broeckx's linearization(KBL), which is the current smallest one in terms of the number of variables and constraints. Our linearization is more efficient when the continuous relaxation is tight. Moreover, the continuous relaxation of our linearization leads to an improvement of the famous Gilmore-Lawler bound(GLB).

This paper is organized as follows. In section 2 we review some canonical linearizations. In section 3 we give our new linearization and its properties. The bound obtained from the continuous relaxation of our linearization is discussed in section 4. A cutting plane method based on the new model is given in section 5. Concluding remarks are made in section 6.

2 The canonical linearizations

There are mainly four LP relaxations of QAP, Lawler's linearization [13] as the first, Kauffmann and Broeckx's linearlization [15], Frieze and Yadegar's linearization [12] and that of Adams and Johnson [1]. Here we only list the standard one given by [1] and the current smallest one (in the sense that it has the smallest number of variables) given by [15].

2.1 *The standard linearization*

Defining new variables $y_{ijkl} = x_{ij}x_{kl}$ results in Adams and Johnson's linearization [1]

$$\min \quad \sum_{i,j}\sum_{k,l} a_{ik}b_{jl}y_{ijkl} \tag{2.1}$$

$$\begin{aligned} s.t. \quad & \sum_i y_{ijkl} = x_{kl}, \quad j,k,l = 1,...,n, \\ & \sum_j y_{ijkl} = x_{kl}, \quad i,k,l = 1,...,n, \\ & y_{ijkl} = y_{klij}, \qquad i,j,k,l = 1,...,n, \\ & y_{ijkl} \geq 0, \qquad i,j,k,l = 1,...,n, \\ & X = (x_{ij})_{n\times n} \in \Pi_n. \end{aligned} \tag{2.2}$$

The above formulation contains n^2 binary variables, n^4 continuous variables and $n^4 + 2n^3 + 2n$ constraints in addition to the nonnegative constraints on the continuous variables. This formulation could be reduced, see [3, 8] and the references therein.

2.2 *The current smallest linearization*

For convenience we make the following assumption.

Assumption 2.1. $a_{ik}b_{jl} \geq 0$ *for all* $i, j, k, l = 1, 2, ..., n$.

This assumption does not lose generality because the coefficient products $a_{ik}b_{jl}$ can be guaranteed nonnegative by adding a sufficiently large constant to all the coefficients a_{ik} and b_{jl} without changing the optimal solution.

Under Assumption 2.1, Kauffmann and Broeckx[15] introduced n^2 new real variables

$$y_{ij} := x_{ij} \sum_{kl} a_{ik}b_{jl}x_{kl}, \quad i, j = 1, ..., n,$$

and substituted them into the objective function (1.1). Then they showed that QAP is equivalent to the following mixed integer linear program (MILP).

$$KBL(A, B) = \quad \min \quad \sum_{i,j} y_{ij} \tag{2.3}$$

$$s.t. \quad u_{ij}x_{ij} + \sum_{k,l} a_{ik}b_{jl}x_{kl} - y_{ij} \leq u_{ij}, \quad i, j = 1, ..., n, \tag{2.4}$$

$$y_{ij} \geq 0, \quad i, j = 1, ..., n, \tag{2.5}$$

$$X = (x_{ij})_{n\times n} \in \Pi_n, \tag{2.6}$$

where the constants u_{ij} satisfy

$$u_{ij} \geq \sum_{k,l} a_{ik}b_{jl}x_{kl} \quad \text{for all } X = (x_{ij})_{n\times n} \in \Pi_n. \tag{2.7}$$

This formulation employs n^2 real variables, n^2 binary variables and $3n^2+2n$ constraints including $2n^2$ nonnegative constraints.

In [15], Kauffmann and Broeckx proved the following result. The proof can also be found in [6].

Theorem 2.1. *Under Assumption 2.1,*

$$QAP(A, B) = KBL(A, B) \ .$$

The constants u_{ij} can be chosen among the following formulas:

$$u_{ij} = \max_{X\in\Pi_n} \sum_{k,l} a_{ik}b_{jl}x_{kl}; \tag{2.8}$$

$$u_{ij} = \sum_{k,l} a_{ik}b_{jl}; \tag{2.9}$$

$$u_{ij} = n \cdot (\max_k a_{ik}) \cdot (\max_l b_{jl}). \tag{2.10}$$

(2.8) is the tightest one, and it seems to require solving n^2 linear assignment problems (LAP). But we can compute (2.8) without solving LAPs, due to the following well-known result of Hardy, Littlewood, and Pólya [14].

Theorem 2.2. *Given two n-dimensional real vectors $a = (a_i)$, $b = (b_i)$ such that $0 \le a_1 \le a_2 \le ... \le a_n$ and $b_1 \ge b_2 \ge ... \ge b_n \ge 0$, the following inequalities hold for any permutation ϕ of $1, 2, ..., n$:*

$$\sum_i a_i b_i \le \sum_i a_i b_{\phi(i)} \le \sum_i a_i b_{n-i+1}. \tag{2.11}$$

It is straightforward to see that the nonnegativity assumption on the vectors can be removed. Define the maximal vector product and the minimal vector product by

$$\langle a, b\rangle_+ = \max_{P\in\Pi_n} \langle a, Pb\rangle, \quad \langle a, b\rangle_- = \min_{P\in\Pi_n} \langle a, Pb\rangle, \tag{2.12}$$

respectively. From Theorem 2.2, one can easily compute the maximal vector product and the minimal vector product of any two vectors. Consequently (2.8) can be obtained because it is exactly the same as

$$u_{ij} = \langle a_{i\cdot}, b_{j\cdot}\rangle_+. \tag{2.13}$$

3 A new linearization

First we rewrite the objective function of QAP (1.1) as

$$f(X) = \sum_{i,j,k,l} a_{ik}b_{jl}x_{ij}x_{kl} = \sum_{i,j}(\sum_{k,l} a_{ik}b_{jl}x_{kl})x_{ij}. \tag{3.1}$$

Our work is based on the following well-known result [2, 18].

Lemma 3.1. *The convex envelope of the bilinear function xy over the domain $[x^L, x^U]\times[y^L, y^U]$ is given by*

$$\max\{x^L y + y^L x - x^L y^L, x^U y + y^U x - x^U y^U\}. \tag{3.2}$$

Directly applying the above lemma, we have:

Corollary 3.1. *For $X = (x_{ij}) \in \Pi_n$*

$$(\sum_{k,l} a_{ik}b_{jl}x_{kl})x_{ij} \ge \max\{l_{ij}x_{ij},\ u_{ij}x_{ij} - u_{ij} + \sum_{k,l} a_{ik}b_{jl}x_{kl}\}, \tag{3.3}$$

where u_{ij} is the corresponding upper bound defined by (2.7), (2.8), (2.9) or (2.10), and the lower bounds defined analogously

$$l_{ij} = \min_{X\in\Pi_n} \sum_{k,l} a_{ik}b_{jl}x_{kl} \;=\; \langle a_{i\cdot}, b_{j\cdot}\rangle_-. \tag{3.4}$$

Actually, as pointed out by a referee, (3.3) can be obtained more easily. Indeed, the left side of the inequality is not less than the first term in the right side by the definition of l_{ij} and it is not less than the second term due to Kauffmann and Broeckx [15].

Thus, by the formulation (3.1) and the above corollary, we can derive the following new linearization.

$$
\begin{aligned}
XYL1(A,B) = \quad \min \quad & \sum_{i,j} y_{ij} && && (3.5)\\
s.t. \quad & y_{ij} \geq l_{ij}x_{ij}, && i,j = 1,2,...n, && (3.6)\\
& y_{ij} \geq u_{ij}x_{ij} - u_{ij} + \sum_{k,l} a_{ik}b_{jl}x_{kl}, && i,j = 1,2,...n, && (3.7)\\
& X = (x_{ij})_{n\times n} \in \Pi_n. && && (3.8)
\end{aligned}
$$

Because $X \in \Pi_n$, we can rewrite (3.1) in the following form

$$
f(X) = \sum_{i,j} \left[\left(\sum_{k\neq i, l\neq j} a_{ik}b_{jl}x_{kl} \right) x_{ij} + a_{ii}b_{jj}x_{ij} \right]. \tag{3.9}
$$

Define

$$
\widetilde{l}_{ij} = \min_{X\in\Pi_n} \sum_{k\neq i, l\neq j} a_{ik}b_{jl}x_{kl} = \langle \widetilde{a}_{i\cdot}, \widetilde{b}_{j\cdot} \rangle_-, \tag{3.10}
$$

$$
\widetilde{u}_{ij} = \max_{X\in\Pi_n} \sum_{k\neq i, l\neq j} a_{ik}b_{jl}x_{kl} = \langle \widetilde{a}_{i\cdot}, \widetilde{b}_{j\cdot} \rangle_+, \tag{3.11}
$$

where $\widetilde{a}_i$ is the vector consisting of the $(n-1)$ components of a_i excluding a_{ii}, and $\widetilde{b}_i$ is defined analogously. Similarly to (3.5)-(3.8), we can obtain another linearization as follows.

$$
\begin{aligned}
XYL2(A,B) = \quad \min \quad & \sum_{i,j} (\widetilde{y}_{ij} + a_{ii}b_{jj}x_{ij}) && && (3.12)\\
s.t. \quad & \widetilde{y}_{ij} \geq \widetilde{l}_{ij}x_{ij}, && i,j = 1,2,...n, && (3.13)\\
& \widetilde{y}_{ij} \geq \widetilde{u}_{ij}x_{ij} - \widetilde{u}_{ij} + \sum_{k\neq i, l\neq j} a_{ik}b_{jl}x_{kl}, && i,j = 1,2,...n, && (3.14)\\
& X = (x_{ij})_{n\times n} \in \Pi_n. && && (3.15)
\end{aligned}
$$

The following lemma follows from Theorem 2.2 and the definitions (3.10), (3.11), (3.4) and (2.8).

Lemma 3.2. *We have the inequalities*

$$
\begin{aligned}
\widetilde{l}_{ij} + a_{ii}b_{jj} \geq l_{ij}, \qquad & i,j = 1,2,...n, && (3.16)\\
\widetilde{u}_{ij} + a_{ii}b_{jj} \leq u_{ij}, \qquad & i,j = 1,2,...n. && (3.17)
\end{aligned}
$$

Furthermore, both sides of inequality (3.16) are equal if and only if a_{ii} is the $k-th$ largest element in a_i and b_{jj} is the $k-th$ smallest one in b_j or vice versa for some k. Similarly, (3.17) holds as equality if and only if both a_{ii} and b_{jj} are the $k-th$ largest (or the $k-th$ smallest) elements in a_i and b_i, respectively, for some k.

Using the above results, we can get the following three propositions.

Proposition 3.1.
$$QAP(A,B) \geq XYL2(A,B). \tag{3.18}$$

Proposition 3.2. *Under Assumption 2.1,*

$$XYL2(A,B) \geq XYL1(A,B). \tag{3.19}$$

Proof It is sufficient to show that $((x_{ij})_{n\times n}, (\widetilde{y}_{ij} + a_{ii}b_{jj}x_{ij})_{n\times n})$ satisfies conditions (3.6)-(3.8) for any feasible point $((x_{ij})_{n\times n}, (\widetilde{y}_{ij})_{n\times n})$ of (3.13)-(3.15).

Let $((x_{ij})_{n\times n}, (\widetilde{y}_{ij})_{n\times n})$ be a feasible point of (3.13)-(3.15), and define

$$y_{ij} = \widetilde{y}_{ij} + a_{ii}b_{jj}x_{ij}, \quad (i,j = 1,2,...n). \tag{3.20}$$

It is easy to see that (3.8) holds as it is the same as (3.15). (3.6) follows directly from (3.16), (3.13) and (3.20). To complete our proof, we only need to prove (3.7). For any given pair $(i,j)(i,j = 1,2,...n)$, we have $x_{ij} = 1$ or $x_{ij} = 0$ due to (3.15). First we assume that $x_{ij} = 1$. In this case (3.14) implies that

$$\begin{aligned} \widetilde{y}_{ij} + a_{ii}b_{jj}x_{ij} &\geq \sum_{k\neq i, l\neq j} a_{ik}b_{jl}x_{kl} + a_{ii}b_{jj}x_{ij} \\ &= \sum_{k,l} a_{ik}b_{jl}x_{kl}, \end{aligned} \tag{3.21}$$

where the last equality in the above relation follows from (3.15) and the fact that $x_{ij} = 1$. (3.21) and (3.20) imply that (3.7) is true provided that $x_{ij} = 1$. Now we assume that $x_{ij} = 0$. In this case, we have $y_{ij} = \widetilde{y}_{ij}$ and (3.7) reduces to

$$y_{ij} \geq -u_{ij} + \sum_{k,l} a_{ik}b_{jl}x_{kl}. \tag{3.22}$$

The fact that u_{ij} is an upper bound for $\sum_{k,l} a_{ik}b_{jl}x_{kl}$ shows that the left hand side of (3.22) cannot be greater than zero. Because $x_{ij} = 0$, (3.13) and (3.20) indicate that $y_{ij} \geq 0$, which implies (3.22). This completes our proof. ■

Proposition 3.3. *Under Assumption 2.1,*

$$XYL1(A,B) \geq KBL(A,B). \tag{3.23}$$

Proof It is easy to see that constraints (2.5) are not stronger than (3.6) when $l_{ij} \geq 0$ and that constraints (3.7) are the same as (2.4). ■

Combining the above three propositions with Theorem 2.1, we immediately have $QAP(A,B) = XYL1(A,B) = XYL2(A,B)$ under Assumption 2.1.

Theorem 3.4. *Under Assumption 2.1,*

$$QAP(A,B) = XYL1(A,B) = XYL2(A,B). \tag{3.24}$$

Note that our new linearizations $XYL1(A,B)$ and $XYL2(A,B)$ have the same size as $KBL(A,B)$. Some stronger properties of $XYL2(A,B)$ are given in the next two sections.

4 A new lower bound

From the above proof of Theorem 3.4, we see that the feasible solutions of $XYL1(A,B)$ ($XYL2(A,B)$) and $KBL(A,B)$ are the same. Now we are going to compare the tightness of their continuous relaxations. To do so we relax the constraint set $(x_{ij})_{n\times n} \in \Pi_n$ to its convex hull, the so-called assignment polytope [4], which is given by

$$S_n = \{(x_{ij})_{n\times n} | \quad \sum_j x_{ij} = 1, \ \sum_i x_{ij} = 1, \quad x_{ij} \geq 0, \ i,j = 1,...,n\}. \tag{4.1}$$

Thus, $RXYL2(A,B)$ is the continuous relaxation of $XYL2(A,B)$:

$$\begin{aligned} RXYL2(A,B) = \quad \min \quad & \sum_{i,j} (\widetilde{y}_{ij} + a_{ii}b_{jj}x_{ij}) && (4.2)\\ s.t. \quad & \widetilde{y}_{ij} \geq \widetilde{l}_{ij}x_{ij}, \quad i,j = 1,2,...n, && (4.3)\\ & \widetilde{y}_{ij} \geq \widetilde{u}_{ij}x_{ij} - \widetilde{u}_{ij} + \sum_{k\neq i, l\neq j} a_{ik}b_{jl}x_{kl}, \quad i,j = 1,2,...n, && (4.4)\\ & X = (x_{ij})_{n\times n} \in S_n. && (4.5) \end{aligned}$$

Similarly $RXYL1(A,B)$ and $RKBL(A,B)$ can be defined.

First we make another assumption which is a little more restrictive than Assumption 2.1 but is still general in practice.

Assumption 4.1. *$a_{ik} \geq 0$, $b_{jl} \geq 0$ for all $i,j,k,l = 1,2,...,n$ and $a_{ii} = b_{jj} = 0$ for all $i,j = 1,2,...,n$.*

Note that the assumption is necessary. Actually, as pointed out by a referee, the following propositions are not generally true under Assumption 2.1. Before presenting the propositions, we firstly give an artificial example.

Example 4.1. *Let*

$$A = \begin{bmatrix} 16 & 8 & 18 \\ 8 & 16 & 18 \\ 18 & 18 & 0 \end{bmatrix}, \ B = \begin{bmatrix} 2 & 3 & 5 \\ 3 & 3 & 5 \\ 5 & 5 & 19 \end{bmatrix}.$$

Then we have

$$\begin{aligned} RKBL(A,B) &= 449 \\ RXYL1(A,B) &= 488 \\ RXYL2(A,B) &= 448. \end{aligned}$$

Proposition 4.1. *Under Assumption 4.1,*

$$RXYL2(A,B) \geq RXYL1(A,B). \tag{4.6}$$

Proof Inequality (4.6) follows directly from Lemma 3.2 and the fact

$$\sum_{k,l} a_{ik}b_{jl}x_{kl} = \sum_{k\neq i, l\neq j} a_{ik}b_{jl}x_{kl} \tag{4.7}$$

which is implied by Assumption 4.1. ■

From Lemma 3.2, we can see that it is very likely that the feasible set of $RXYL2(A, B)$ is smaller than that of $RXYL1(A, B)$. Therefore it is natural for us to expect that in most cases (4.6) holds as an inequality, namely

$$RXYL2(A, B) > RXYL1(A, B). \tag{4.8}$$

Due to the above proposition and the fact that both $RXYL1(A, B)$ and $RXYL2(A, B)$ have the same computational complexity, in the following we only discuss $RXYL2(A, B)$. Similarly to Proposition 3.3 and Proposition 4.1, we have the following result.

Proposition 4.2. *Under Assumption 4.1,*

$$RXYL2(A, B) \geq RKBL(A, B). \tag{4.9}$$

Inequality (4.9) seldom holds as equality. Namely it is often the case

$$RXYL2(A, B) > RKBL(A, B), \tag{4.10}$$

which is demonstrated by our numerical results given in Table 1. The test problems are taken from QAPLIB [10]. From these limited results, it seems that the relaxation $RKBL(A, B)$ is extremely weak, which normally gives nearly zero objective function values. This also explains why $RKBL(A, B)$ has not been used in the literatures. In Table 1, the bounds obtained by $RXYL2(A, B)$ have been rounded to the corresponding minimal upper integers since the elements in A and B are integers for all the tested problems. It is encouraging to see that in 7 out of the 8 problems the bounds derived by our relaxation are better than the bounds derived by the famous Gilmore-Lawler relaxation [13]:

$$GLB(A, B) \quad = \quad \min \sum_{i,j} (\widetilde{l}_{ij} + a_{ii}b_{jj})x_{ij}, \tag{4.11}$$

$$s.t. \quad X = (x_{ij})_{n\times n} \in \Pi_n, \tag{4.12}$$

where $\widetilde{l}_{ij}$ are defined by (3.10). Some theoretical analysis on the relations between our relaxation and the Gilmore-Lawler relaxation is given below.

Table 1. Comparison among GLB, $RXYL2$ and $RKBL$

Prob.	GLB	$RXYL2$	$RKBL$	QAP
chr12a	7245	7457	3.3737e-16	9552
chr12b	7146	7300	1.5531e-11	9742
chr18a	6779	6885	7.7394e-14	11098
chr18b	1534	1534	2.8136e-13	1534
had14	2492	2494	5.6831e-12	2724
rou12	202272	203215	1.3209e-12	235528
rou15	298548	298956	1.2041e-15	354210
tai12a	195918	196981	6.4164e-16	224416

We observe that $XYL2(A, B)$ (also $RXYL2(A, B)$) is exactly the canonical Gilmore-Lawler bound (GLB) if we delete the constraints (3.14) from $XYL2(A, B)$. The following theorem was first presented in [21].

Theorem 4.3. *We have*

$$RXYL2(A,B) \geq GLB(A,B). \tag{4.13}$$

Furthermore, we have that

$$RXYL2(A,B) > GLB(A,B) \tag{4.14}$$

if $QAP \neq GLB(A,B)$ and $GLB(A,B)$ has a unique optimal solution.

Proof We only need to prove the second result since (4.13) is obvious. Let $X^* = \{x^*_{ij}\}$ be the unique optimal solution of $GLB(A,B)$. Define $\widetilde{Y}^*$ by

$$\widetilde{y}^*_{ij} = \widetilde{l}_{ij} x^*_{ij}. \tag{4.15}$$

Thus, $(X^*, \widetilde{Y}^*)$ is also a feasible solution of (3.12), (3.13) and (3.15). Define the set $\Delta = \{(i,j) \mid x^*_{ij} = 1\}$. If

$$\widetilde{l}_{ij} \geq \sum_{k \neq i, l \neq j} a_{ik} b_{jl} x^*_{kl}, \quad \forall (i,j) \in \Delta, \tag{4.16}$$

it follows from the definition of $\widetilde{l}_{ij}$ that

$$\widetilde{l}_{ij} = \sum_{k \neq i, l \neq j} a_{ik} b_{jl} x^*_{kl}, \quad \forall (i,j) \in \Delta. \tag{4.17}$$

Thus, we have that

$$\sum_{i,j} \Big(\sum_{k \neq i, l \neq j} a_{ik} b_{jl} x^*_{kl} + a_{ii} b_{jj} \Big) x^*_{ij} = \sum_{i,j} (\widetilde{l}_{ij} + a_{ii} b_{jj}) x^*_{ij}, \tag{4.18}$$

i.e., $QAP = GLB(A,B)$, which is a contradiction to our assumption $QAP \neq GLB(A,B)$. Thus there exists $(i,j) \in \Delta$ for which the inequality (4.16) does not hold, which implies that

$$\widetilde{y}^*_{ij} = \widetilde{l}_{ij} < \sum_{k \neq i, l \neq j} a_{ik} b_{jl} x^*_{kl}. \tag{4.19}$$

The above inequality contradicts (3.14). Therefore we have proved that $(X^*, \widetilde{Y}^*)$ is not a feasible point of $RXYL2(A,B)$. Consider the continuous relaxation problem

$$\begin{aligned} RGLB(A,B) \quad = \quad \min \ & \sum_{i,j} (\widetilde{y}_{ij} + a_{ii} b_{jj} x_{ij}) && (4.20) \\ s.t. \ & \widetilde{y}_{ij} \geq \widetilde{l}_{ij} x_{ij}, \qquad i,j = 1,2,...n\ , && (4.21) \\ & X = (x_{ij})_{n \times n} \in S_n. && (4.22) \end{aligned}$$

It is easy to see that $(X^*, \widetilde{Y}^*)$ is the unique optimal solution of (4.20)-(4.22). But since $(X^*, \widetilde{Y}^*)$ is not a feasible point of $RXYL2(A,B)$, it follows that $RGLB(A,B) < RXYL2(A,B)$. Consequently (4.14) follows from the fact that $RGLB(A,B) = GLB(A,B)$. ■

In [21], Lagrangian relaxations of $XYL2(A,B)$ were also discussed. Moreover, a fuzzy bound was also presented there and shown to achieve excellent numerical performance, though it is still open whether it is a true bound for the QAP.

5 A cutting plane method based on the new model

In some sense $XYL1(A,B)$ is an improvement of $KBL(A,B)$. So the algorithm [9, 15] based on Benders' decomposition approach can also be used here to solve $XYL1(A,B)$ almost directly. But the result should be more promising. We show this in detail.

Problem (3.5) can be decomposed as

$$\min_{X\in\Pi_n} \left(\min_{y\in Y(x)} \sum_{i,j} y_{ij} \right) \tag{5.1}$$

with

$$Y(x) := \{y \in \Re^{n^2} |\ y_{ij} \geq l_{ij}x_{ij},\ y_{ij} \geq u_{ij}x_{ij} - u_{ij} + \sum_{k,l} a_{ik}b_{jl}x_{kl},\ i,j = 1,2,...n\}. \tag{5.2}$$

For fixed x we dualize the constraints of the second-stage problem $\min_{y\in Y(x)} \sum_{i,j} y_{ij}$, and denote the dual variables corresponding to the second constraints by λ_{ij} $(i,j = 1,2,...n)$. Then we get the subproblem

$$SP(X): \quad \max \quad \sum_{i,j}(\sum_{k,l} a_{ik}b_{jl}x_{kl} - u_{ij} + u_{ij}x_{ij} - l_{ij}x_{ij})\lambda_{ij} + \sum_{i,j} l_{ij}x_{ij} \tag{5.3}$$

$$s.t. \quad 0 \leq \lambda_{ij} \leq 1, \qquad i,j = 1,2,...n. \tag{5.4}$$

It can easily be checked that

$$\lambda_{ij} := \widetilde{x}_{ij}, \qquad i,j = 1,2,...n, \tag{5.5}$$

is an optimal solution of the subproblem $SP(\widetilde{X})$ because of the definition of the constants u_{ij} and l_{ij} and the constraints $0 \leq \lambda_{ij} \leq 1$. The feasible solution set F of $SP(X)$ does not depend on the chosen vector X. Therefore let $\lambda^{(t)}$ be the incidence vectors of the extreme points of F (which is the unit hypercube in $\Re^{n^2}$). Introducing

$$c_{ij}^{(t)} := \sum_{k,l} \lambda_{kl}^{(t)} a_{ki}b_{lj} + \lambda_{ij}^{(t)}u_{ij} - \lambda_{ij}^{(t)}l_{ij} + l_{ij}, \tag{5.6}$$

$$\alpha^{(t)} := \sum_{i,j} \lambda_{ij}^{(t)}u_{ij}, \qquad t = 1,2,...,2^{n^2} =: T, \tag{5.7}$$

we can see that problem (5.1) is equivalent to

$$\min_{X\in\Pi_n} \max_{1\leq t\leq T} \{\sum_{i,j} c_{ij}^{(t)}x_{ij} - \alpha^{(t)}\} \tag{5.8}$$

by the fact that for any fixed x the second-stage problem $\min_{y\in Y(x)} \sum_{i,j} y_{ij}$ of (5.1) is a linear programming whose dual is just (5.3)-(5.4) and the fact that the linear programming (5.3)-(5.4) has an optimal solution at an extreme point of F. Problem (5.8) yields now the master program

$$MP: \quad \min \quad z \tag{5.9}$$

$$s.t. \quad z \geq \sum_{i,j} c_{ij}^{(t)}x_{ij} - \alpha^{(t)}, \quad 1 \leq t \leq T, \tag{5.10}$$

$$X = (x_{ij})_{n\times n} \in \Pi_n. \tag{5.11}$$

As in any decomposition approach the master problem is not solved for all restrictions $z \geq \sum_{i,j} c_{ij}^{(t)} x_{ij} - \alpha^{(t)}$, $(1 \leq t \leq T)$, but only for a subset $\{t | \; 1 \leq t \leq r\}$ of indices. We denote this restricted master problem by $MP(r)$. Getting an optimal solution $\widetilde{X}$ for this restricted master problem, the subproblem $SP(\widetilde{X})$ is solved, which yields $\lambda_{ij}^{(r+1)} := \widetilde{x}_{ij}$ and a new constraint

$$z \geq \sum_{i,j} c_{ij}^{(r+1)} x_{ij} - \alpha^{(r+1)} \tag{5.12}$$

is added to the current $MP(r)$, which yields $MP(r+1)$. Then we have the following result.

Proposition 5.1. *Assume that $\widetilde{X}$ is an optimal solution of $MP(r)$ and $\lambda_{ij}^{(r+1)} := \widetilde{x}_{ij}$. For any $s > r$, $\widetilde{X}$ cannot be an optimal solution of $MP(s)$ unless it is the optimal solution of QAP.*

Proof Denote the optimal objective function value of any master problem $MP(s)$ by $\widetilde{z}_s$, which is a lower bound for QAP. If $\widetilde{X}$ is also an optimal solution of $MP(s)$ for some $s > r$, it follows that

$$\widetilde{z}_s \geq \sum_{i,j} c_{ij}^{(r+1)} \widetilde{x}_{ij} - \alpha^{(r+1)}, \tag{5.13}$$

since $MP(s)$ contains the constraint (5.12). The left-hand side of (5.13) is a lower bound for QAP while the right-hand side of (5.13) corresponds to a feasible objective function value of QAP ($f(\widetilde{X})$ of (1.1)), which can be shown as follows:

$$\begin{aligned}
&\sum_{i,j} c_{ij}^{(r+1)} \widetilde{x}_{ij} - \alpha^{(r+1)} \\
&= \sum_{i,j} (\sum_{k,l} \lambda_{kl}^{(r+1)} a_{ki} b_{lj} + \lambda_{ij}^{(r+1)} u_{ij} - \lambda_{ij}^{(r+1)} l_{ij} + l_{ij}) \widetilde{x}_{ij} - \sum_{i,j} \lambda_{ij}^{(r+1)} u_{ij} \\
&= \sum_{i,j} (\sum_{k,l} \widetilde{x}_{kl} a_{ki} b_{lj} + \widetilde{x}_{ij} u_{ij} - \widetilde{x}_{ij} l_{ij} + l_{ij}) \widetilde{x}_{ij} - \sum_{i,j} \widetilde{x}_{ij} u_{ij} \\
&= \sum_{i,j} \sum_{k,l} a_{ki} b_{lj} \widetilde{x}_{kl} \widetilde{x}_{ij}.
\end{aligned}$$

Therefore (5.13) holds as equality and $\widetilde{X}$ must be the optimal solution of QAP. ■

From the above proposition, an optimal solution of QAP will be obtained in a finite number of steps.

Note that if we set $l_{ij} = 0$ for all $i, j = 1, 2, ..., n$, then $XYL1(A, B)$ is just the same as $KBL(A, B)$. Thus it is easy to see the inner objective function of (5.8) or the right-hand side of (5.10) satisfies

$$\begin{aligned}
&\sum_{i,j} c_{ij}^{(t)} x_{ij} - \alpha^{(t)} \\
&= \sum_{i,j} (\sum_{k,l} \lambda_{kl}^{(t)} a_{ki} b_{lj} + \lambda_{ij}^{(t)} u_{ij}) x_{ij} - \sum_{i,j} \lambda_{ij}^{(t)} u_{ij} + \sum_{i,j} l_{ij} (1 - \lambda_{ij}^{(t)}) x_{ij} \\
&\geq \sum_{i,j} (\sum_{k,l} \lambda_{kl}^{(t)} a_{ki} b_{lj} + \lambda_{ij}^{(t)} u_{ij}) x_{ij} - \sum_{i,j} \lambda_{ij}^{(t)} u_{ij},
\end{aligned} \tag{5.14}$$

which is the corresponding part in the case of $KBL(A, B)$. And (5.14) becomes an equality if and only if $\lambda_{ij}^{(t)} = x_{ij}$ for all $i, j = 1, 2, ..., n$ under the assumption $l_{ij} > 0$. Finally we have the following result.

Theorem 5.2. *Assume $l_{ij} > 0$ for all $i, j = 1, 2, ..., n$. The master program $MP(r)$ based on $XYL1(A, B)$ gives a lower bound strictly better than the corresponding $MP(r)$ based on $KBL(A, B)$ until the algorithm stops.*

Proof If the algorithm has not stopped at step r, the optimal solution $\widetilde{X}$ must be different from $\lambda^{(t)}$ for any $1 \leq t \leq r$. From the above analysis, we know (5.14) is a strict inequality for any $1 \leq t \leq r$. Note that $MP(r)$ based on $KBL(A, B)$ is just the case $l_{ij} = 0$ of $MP(r)$ based on $XYL1(A, B)$. Therefore the objective function value of $MP(r)$ based on $XYL1(A, B)$ is strictly larger than that based on $KBL(A, B)$. ■

However, solving the master problem $MP(r)$ optimally is in general as difficult as solving QAP.

To our interest, we can exactly solve QAP in this cutting plane framework by getting a little weaker lower bound from solving the continuous relaxation of $MP(r)$ based on $XYL1(A, B)$, which is a linear program. This cannot be done in the framework based on $KBL(A, B)$, because the corresponding bound is too weak as shown in Table 1.

Instead of solving $MP(r)$ exactly, we can also try to find a suboptimal solution for $MP(r)$ at which the objective function value is strictly less than that of the current best solution of the QAP [9]. As in [7], which is an improvement of [5, 9], the following heuristic is proposed to find such a suboptimal solution. First solve LAP

$$\lambda^{(r)} := \arg \min_{(\lambda_{ij}) \in \Pi_n} \sum_{ij} \sum_{i,j} c_{ij}^{(r)} \lambda_{ij}. \tag{5.15}$$

Define

$$\beta^{(r)} := \max \left(1, \ |\sum_{i,j} c_{ij}^{(r)} \lambda_{ij}^{(r)} - \alpha^{(r)}|\right), \tag{5.16}$$

and a new direction h of search with elements

$$h_{ij}^{(r)} := h_{ij}^{(r-1)} + \frac{1}{\beta^{(r)}} c_{ij}^{(r)}, \quad i, j = 1, 2, ..., n, \tag{5.17}$$

with the initial value $h_{ij}^{0} := 0, i, j = 1, 2, ..., n$. Then the following LAP, which is an approximation for $MP(r)$ is solved

$$\min_{(x_{ij})_{n \times n} \in \Pi_n} \sum_{ij} h_{ij}^{(r)} x_{ij}. \tag{5.18}$$

This solution can still be improved with respect to the objective function value of the given QAP by applying pair and triple exchange algorithms. More formally we propose the following algorithm:

Algorithm 5.1.

Step 1 *Initialize* $t := 1$, $h_{ij}^0 := 0, i, j = 1, 2, ..., n$. *Input an integer* $MITER > 0$. *Compute* u_{ij} *and* l_{ij} $(i, j = 1, 2, ..., n)$ *by sorting* A *and* B *and applying Theorem 2.2. Start with feasible* $(x_{ij}^{(1)}) \in \Pi_n$ *and the corresponding objective function value* z^* *of the QAP.*

Step 2 *Compute (5.6) and (5.7) for* $i, j = 1, 2, ..., n$.

Step 3 *Solve (5.15) and compute (5.16) and (5.17).*

Step 4 *Solve (5.18). Let* x^{t+1} *be the solution of this problem and* z^{t+1} *the corresponding objective function value of the QAP.*

Step 5 *If* $z^{t+1} < z^*$, *put* $z^* := z^{t+1}$

Step 6 *If* $t < MITER$, *replace t by t+1 and go to 2; else stop.* z^* *is the best objective function value found during the procedure.*

Steps 1-5 of the algorithm are iterated by a fixed number (MITER) of times. Similarly to [7], we repeat the algorithm by starting from randomly generated feasible solutions. The number of restarts is denoted by 'REP'.

The above procedure is also suitable for $XYL2(A, B)$ and it seems better to use $XYL2(A, B)$ instead of $XYL1(A, B)$ from analogous analysis in the previous section. Furthermore in many cases where $a_{ii} = b_{ii} = 0$ for all $i = 1, 2, ..., n$, the difference between $XYL2(A, B)$ and $XYL1(A, B)$ is just in the difference of the values of l_{ij} and u_{ij}. As showed in [7], different u_{ij} does not have great influence on the performance of the algorithm. Thus we are only interested in the above algorithm based on $XYL2(A, B)$, denoted by HXYL. We denote the algorithm [7] based on $KBL(A, B)$ by HKBL. The only difference is in the additional storage for $\widetilde{l}_{ij}$ in HXYL because the computational complexity is almost the same among (2.10), (2.13), (3.4), (3.10) and (3.11). Note that HKBL uses (2.10) while HXYL uses (3.10) and (3.11). As in [7] we set the parameters MITER= 15 and REP= $3n$.

Table 2. Comparison between HKBL and HXYL

	HKBL				HXYL			
Prob.	A(%)	B(%)	C(%)	CPU(s)	A(%)	B(%)	C(%)	CPU(s)
lipa20a	0.07	0.71	0	1.88	0	0	0	1.94
nug30	0.39	0.69	0.001	10.43	0.16	0.42	0	9.86
kra30b	0.30	0.62	0	9.97	0.21	0.40	0	9.74
tho40	0.54	0.93	0.27	35.86	0.38	0.70	0.11	33.99
sko42	0.49	0.66	0.18	44.10	0.29	0.43	0.09	41.57
sko49	0.53	0.84	0.37	88.70	0.35	0.50	0.12	78.97
wil50	0.22	0.29	0.18	97.59	0.13	0.16	0.09	93.99
esc64a	0	0	0	156.81	0	0	0	147.66
sko81	0.51	0.65	0.42	782.81	0.32	0.41	0.24	708.81

We tested several examples from QAPLIB [10]. As in [7], each example has been tested by a series of 10 independent runs. Numerical results are reported in Table 2, where column A gives the average deviation, column B shows the worst deviation, and column C gives the best reached deviation of the 10 tests. The given average running CPU time in seconds is obtained using a CPU P4 with 2.4GHz. Table 2 shows that HXYL could usually find better solutions than HKBL in less CPU time. Especially for problem nug30, HXYL finds the optimal solution while HKBL cannot.

6 Conclusions and further work

In this paper, we have given new linearizations ($XYL1$, $XYL2$), which have the same size as KBL and are more efficient in terms of the tightness of the continuous relaxation. Furthermore, the continuous relaxation of $XYL2$ can be regarded as an improvement of the Gilmore-Lawler bound (GLB). It is the next work to answer whether we could get better result by combining this new bound with the branch-and-bound method. We also give a corresponding cutting plane heuristic method as in [7] and show its superiority. This heuristic could be further improved in two directions, combining with Tabu search or providing initial values for other heuristic methods such as simulated annealing.

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