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Componentwise Error Bounds for Linear Complementarity Problems

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Componentwise error bounds for linear complementarity problems are presented. For the problem with an H-matrix, the error bound can be computed by solving a system of linear equations. It is proved that our error bound is more accurate than that obtained recently by Chen and Xiang [Math. Prog., Ser. A 106, 513-525 (2006)]. Numerical results show that the new bound is often much better than previous ones.

Keywords: error bound, linear complementarity problem, convex quadratic programming

1. Introduction

Let $M \in \Re^{n\times n}$ and $q \in \Re^n$ be given. The linear complementarity problem (LCP), denoted by $LCP(M,q)$, is to find a vector $x^* \in \Re^n$ such that

$$x^* \geqslant 0, \qquad Mx^* + q \geqslant 0, \qquad (x^*)^T(Mx^*+q) = 0,$$

or to show that no such x^* exists. Here the inequalities are meant componentwise. LCPs have many important real world applications, for example, see Cottle, Pang & Stone (1992) and Ferris & Pang

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(1997).

Let $\hat{x} \in \Re^n_+$ be an arbitrary but fixed vector. Estimation of the error $\hat{x} - x^*$ plays an important role in both numerical solution and theoretical analysis. Norm error estimation for LCPs has been extensively studied so far, for example, see Chen & Xiang (2006), Mangasarian & Ren (1994), Mathias & Pang (1990) and Pang (1997).

We present in this paper new componentwise error bounds. That is, we give $r \in \Re^n_+$ such that

$$|\hat{x} - x^*| \leqslant r,$$

where notation $|y|$ means the vector whose $i-$th component is $|y_i|$. Such an r can be computed in general by finding a feasible solution of a convex quadratic programming problem, for which a very mild computational cost is expected. The solution of the convex quadratic programming delivers a tight error bound.

For the case that M is an H-matrix, that is, the comparison matrix $\bar{M} = (\bar{m}_{ij}) \in \Re^{n\times n}$ has a nonnegative inverse, where

$$\bar{m}_{ij} = \begin{cases} |m_{ii}| & \text{if} \quad i = j, \\ -|m_{ij}| & \text{if} \quad i \neq j, \end{cases}$$

we present a tight error bound $\varphi_H(\hat{x})$, which can be computed by solving a system of linear equations. For LCP with H-matrix, the error bound

$$\|\hat{x} - x^*\|_p \leqslant \|\bar{M}^{-1} \max(D, I)\|_p \|\min(\hat{x}, M\hat{x} + q)\|_p =: \phi_p(\hat{x}) \tag{1.1}$$

was given by Chen & Xiang (2006), where $p \in [1, +\infty]$, D is the diagonal part of M, I is the identity matrix, "max" and "min" are taken componentwise. The error bound $\phi_p(\hat{x})$ was proved more accurate than the well known bound given by Mathias & Pang (1990). We will prove that our error bound $\varphi_H(\hat{x})$ is more accurate than the error bound $\phi_p(\hat{x})$ given in (1.1) in the sense that

$$\|\varphi_H(\hat{x})\|_p \leqslant \phi_p(\hat{x}).$$

The numerical results illustrate that this inequality holds strictly, and in different order of magnitude for our model problem.

To conclude this section we give some notations. The set

$$FEA(M, q) := \{x \in \Re^n : x \geqslant 0, Mx + q \geqslant 0\} \tag{1.2}$$

is called the *feasible set* of the linear complementarity problem $LCP(M, q)$. An element of $FEA(M, q)$ is called a *feasible vector* of the problem $LCP(M, q)$. This feasible vector is also known as a *feasible solution*.

For $x = (x_i)$ and $y = (y_i) \in \Re^n$, $x \leqslant y$ stands for $x_i \leqslant y_i$, $i = 1, \ldots, n$. We denote by $\max(x, y)$ and by $\min(x, y)$ the componentwise maximum and minimum of x and y, respectively. Let $\underline{a} = (\underline{a}_i) \in \Re^n$ and $\overline{a} = (\overline{a}_i) \in \Re^n$ be given with $\underline{a} \leqslant \overline{a}$. We define an n-dimensional interval vector as the set of vectors

$$[a] = [\underline{a}, \overline{a}] := \{x \in \Re^n : \underline{a} \leqslant x \leqslant \overline{a}\}.$$

For an interval vector $[a] = [\underline{a}, \overline{a}]$, we define the operation

$$\max(0, [a]) := [\max(0, \underline{a}), \max(0, \overline{a})].$$

For simplicity, we write $[a, a] = a$ for $a \in \Re^n$. The operations $+$, $-$, and $\times$ can be defined for intervals. We refer to Neumaier (1990) for details.

2. Error Bound

We begin our study with an existence theorem. It gives a sufficient condition for guaranteeing that an interval vector contains a solution of an LCP.

THEOREM 2.1 Let $M \in \Re^{n\times n}$ and $q \in \Re^n$ be given. Let $[x]$ be an n-dimensional interval vector, and $\hat{x} \in [x]$ be an arbitrary but fixed vector. Let a diagonal matrix $\Delta = diag(\delta_i) \in \Re^{n\times n}$ be given with $\delta_i > 0$, $i = 1,\dots,n$. If

$$\Gamma(\hat{x},[x],\Delta) \;:=\; \max(0, \hat{x} - \Delta(M\hat{x}+q) + (I-\Delta M)([x]-\hat{x})) \;\subseteq\; [x], \tag{2.1}$$

then $LCP(M,q)$ has a solution $x^* \in \Gamma(\hat{x},[x],\Delta)$.

Proof. See Alefeld, Wang & Shen (2004). □

Now we find a vector $r \in \Re^n_+$ such that the condition (2.1) holds for the interval vector $[\hat{x}-r, \hat{x}+r]$. Consequently, it follows from Theorem 2.1 that $LCP(M,q)$ has a solution $x^* \in [\hat{x}-r,\hat{x}+r]$, which implies the componentwise error bound

$$|\hat{x} - x^*| \leqslant r.$$

Given a vector $\hat{x} \in \Re^n_+$, we define

$$\alpha \quad := \quad \{i : \hat{x}_i \leqslant (M\hat{x}+q)_i\}, \tag{2.2}$$

$$\tilde{x} \quad := \quad (\tilde{x}_i) \quad \text{with} \quad \tilde{x}_i = \begin{cases} \hat{x}_i & \text{if } i \in \alpha, \\ 0 & \text{if } i \notin \alpha, \end{cases} \tag{2.3}$$

$$\tilde{y} \quad := \quad (\tilde{y}_i) \quad \text{with} \quad \tilde{y}_i = \begin{cases} (M\hat{x}+q)_i & \text{if } i \in \alpha, \\ -|(M\hat{x}+q)_i| & \text{if } i \notin \alpha, \end{cases} \tag{2.4}$$

$$\tilde{M} \quad := \quad D - |B|, \tag{2.5}$$

$$\tilde{q} \quad := \quad \tilde{M}\tilde{x} + \tilde{y}, \tag{2.6}$$

where D and $-B$ are the diagonal and the off-diagonal parts of M, respectively.

Using these definitions, we can give our main error estimation result as follows.

THEOREM 2.2 Let $M \in \Re^{n\times n}$, $q \in \Re^n$ and $\hat{x} \in \Re^n_+$ be given. Let $\tilde{M} \in \Re^{n\times n}$ and $\tilde{q} \in \Re^n$ be defined by (2.5) and (2.6), respectively. If $u \in FEA(\tilde{M},\tilde{q})$, then $LCP(M,q)$ has a solution $x^* \in [x] := [\hat{x}-r, \hat{x}+r]$, where $r = \tilde{x}+u$, $\tilde{x} \in \Re^n$ is defined by (2.3). As a direct consequence, we have the error bound

$$|\hat{x} - x^*| \;\leqslant\; r. \tag{2.7}$$

Proof. It is sufficient to show that condition (2.1) holds for $r = \tilde{x} + u$. We define a positive definite diagonal matrix $\Delta = diag(\delta_i) \in \Re^{n\times n}$ by setting

$$\delta_i = \begin{cases} 1 & \text{if} \quad m_{ii} \leqslant 0, \\ 1/m_{ii} & \text{if} \quad m_{ii} > 0. \end{cases}$$

It can be verified that (see Neumaier (1990))

$$(I - \Delta M)([x] - \hat{x}) \;=\; [-|I - \Delta M|r, |I - \Delta M|r].$$

By the definition of $\Gamma(\hat{x},[x],\Delta)$, we have $\Gamma(\hat{x},[x],\Delta) = [\underline{\Gamma(\hat{x},[x],\Delta)}, \overline{\Gamma(\hat{x},[x],\Delta)}]$, where

$$\begin{aligned} \underline{\Gamma(\hat{x},[x],\Delta)} &= \max(0, \hat{x} - \Delta(M\hat{x} + q) - |I - \Delta M|r), \\ \overline{\Gamma(\hat{x},[x],\Delta)} &= \max(0, \hat{x} - \Delta(M\hat{x} + q) + |I - \Delta M|r). \end{aligned}$$

Firstly, we prove $\overline{\Gamma(\hat{x},[x],\Delta)} \leqslant \hat{x} + r$. Remembering that D and $-B$ are the diagonal and off-diagonal parts of M respectively, we have

$$|I - \Delta M|r \;=\; r - \Delta(D - |B|)r \;=\; r - \Delta\tilde{M}r.$$

The definition (2.4) yields $M\hat{x} + q \geqslant \tilde{y}$. Note that $\tilde{M}\tilde{x} + \tilde{y} = \tilde{q}$, $r = \tilde{x} + u$, and $\tilde{M}u + \tilde{q} \geqslant 0$. Thus, it follows that

$$\begin{aligned} \hat{x} - \Delta(M\hat{x} + q) + |I - \Delta M|r &\leqslant \hat{x} - \Delta\tilde{y} + r - \Delta\tilde{M}r \\ &\leqslant \hat{x} + r - \Delta\tilde{y} - \Delta\tilde{M}(\tilde{x} + u) \\ &= \hat{x} + r - \Delta(\tilde{M}u + \tilde{M}\tilde{x} + \tilde{y}) \\ &= \hat{x} + r - \Delta(\tilde{M}u + \tilde{q}) \\ &\leqslant \hat{x} + r, \end{aligned}$$

which, together with $\hat{x} + r = \hat{x} + \tilde{x} + u \geqslant u \geqslant 0$, yields $\overline{\Gamma(\hat{x},[x],\Delta)} \leqslant \hat{x} + r$.

Now we prove $\underline{\Gamma(\hat{x},[x],\Delta)} \geqslant \hat{x} - r$. For any index $i \in \alpha$, we have

$$[\underline{\Gamma(\hat{x},[x],\Delta)}]_i \;=\; 0 \;\geqslant\; -u_i \;=\; \hat{x}_i - r_i.$$

For any index $i \notin \alpha$, we note from the definition (2.4) that $-(M\hat{x} + q)_i \geqslant \tilde{y}_i$. Considering that $\tilde{M}\tilde{x} + \tilde{y} = \tilde{q}$, $u \in FEA(\tilde{M}, \tilde{q})$ and that $r = \tilde{x} + u$, we have

$$\begin{aligned} [\underline{\Gamma(\hat{x},[x],\Delta)}]_i &\geqslant [\hat{x} - \Delta(M\hat{x} + q) - |I - \Delta M|r]_i \\ &\geqslant [\hat{x} + \Delta\tilde{y} - r + \Delta\tilde{M}r]_i \\ &\geqslant \hat{x}_i - r_i + [\Delta\tilde{y} + \Delta\tilde{M}(\tilde{x} + u)]_i \\ &= \hat{x}_i - r_i + [\Delta(\tilde{M}u + \tilde{M}\tilde{x} + \tilde{y})]_i \\ &= \hat{x}_i - r_i + [\Delta(\tilde{M}u + \tilde{q})]_i \\ &\geqslant \hat{x}_i - r_i. \end{aligned}$$

Hence, it is shown that condition (2.1) holds for $[x] := [\hat{x} - r, \hat{x} + r]$ with $r = \tilde{x} + u$. Therefore, it follows from Theorem 2.1 that $LCP(M,q)$ has a solution $x^* \in [x] := [\hat{x} - r, \hat{x} + r]$, which in turns implies that the error bound (2.7) holds. This completes our proof. □

We note that the matrix $\tilde{M}$, defined by (2.5), is a Z-matrix. A Z-matrix is a matrix whose all off-diagonal entries are non-positive. So, if $FEA(\tilde{M},\tilde{q}) \neq \emptyset$, there is a unique vector $u^* \in FEA(\tilde{M},\tilde{q})$, which is a solution of $LCP(\tilde{M},\tilde{q})$, such that for any $u \in FEA(\tilde{M},\tilde{q})$ we have $u^* \leqslant u$ (Cottle, Pang & Stone, 1992, p.198–212). This vector u^* is usually called the *least element* of the feasible set $FEA(\tilde{M},\tilde{q})$. As u^* is also a feasible vector, we obtain the following error estimation result, which is a special case of Theorem 2.2.

COROLLARY 2.1 In the setting of Theorem 2.2, we let u^* be the least element of $FEA(\tilde{M},\tilde{q})$, which is unique. Define

$$\varphi(\hat{x}) := \tilde{x} + u^*. \tag{2.8}$$

Then $LCP(M,q)$ has a solution $x^* \in [x] := [\hat{x} - \varphi(\hat{x}), \hat{x} + \varphi(\hat{x})]$. As a direct consequence, we have the error estimate

$$|\hat{x} - x^*| \leqslant \varphi(\hat{x}). \tag{2.9}$$

Corollary 2.1 indicates that the error bound $\varphi(\hat{x})$, defined by the least element, is the "sharpest" comparing with the one defined only by a certain feasible vector of $LCP(\tilde{M},\tilde{q})$. To compute $\varphi(\hat{x})$, we need find the least element u^* of the feasible set $FEA(\tilde{M},\tilde{q})$. It can be cast into the following convex quadratic programming:

$$\begin{aligned} \min \quad & u^T u \\ s.t. \quad & \tilde{M}u + \tilde{q} \geqslant 0 \\ & u \geqslant 0, \end{aligned} \tag{2.10}$$

which can be solved efficiently by existing software, for example, CVX by Grant, Boyd & Ye (2008). Of course, sometimes we do not need to solve the problem (2.10) exactly, because any feasible solution of (2.10) provides an error bound (2.7). In this way, a much smaller computational cost can be expected.

Now, we consider the special case that M is an H-matrix whose all diagonal elements are positive. An H-matrix is a matrix whose comparison matrix is an M-matrix, while an M-matrix is a matrix whose all off-diagonal elements are non-positive and whose inverse has no negative elements (see Plemmons (1977)). In this case, the problem $LCP(M,q)$ has a unique solution for any $q \in \Re^n$ (see Cottle, Pang & Stone (1992), p.148–152). LCP with an H-matrix appears frequently in modeling real world problems, see, e.g., Ferris & Pang (1997) and Rodrigues (1987). The following theorem shows that for such an LCP, an error bound given by (2.7) can be obtained by solving a system of linear equations.

THEOREM 2.3 Suppose that M is an H-matrix whose diagonal elements are all positive. Let $\tilde{x}$, $\tilde{M}$ and $\tilde{q}$ be defined by (2.3), (2.5) and (2.6), respectively. Then we have the estimate

$$|\hat{x} - x^*| \leqslant \varphi_H(\hat{x}) := \tilde{x} + \tilde{M}^{-1} \max(0, -\tilde{q}), \tag{2.11}$$

where x^* is the unique solution of $LCP(M,q)$.

REMARK 2.1 The estimate (2.11) can be computed by solving a system of linear equations.

Proof. Since M is assumed to be an H-matrix whose all diagonal elements are positive, we have

$\tilde{M}^{-1} \geqslant 0$, that is, each element of $\tilde{M}^{-1}$ is nonnegative. So

$$\begin{aligned} u &= \tilde{M}^{-1}\max(0,-\tilde{q}) \geqslant 0 \\ \tilde{M}u+\tilde{q} &= \tilde{M}\tilde{M}^{-1}\max(0,-\tilde{q})+\tilde{q} = \max(0,-\tilde{q})+\tilde{q} \geqslant 0. \end{aligned}$$

It means $u = \bar{M}^{-1}\max(0,-\tilde{q}) \in FEA(\tilde{q},\tilde{M})$. Therefore, (2.11) follows from (2.7) in Theorem 2.2. □

We mentioned in the Introduction that the error bound (1.1) was given by Chen & Xiang (2006) for the LCP with an H-matrix. This bound was proved to be more accurate than the well known error bound given by Mathias & Pang (1990). Now, we show that our error bound (2.11) is more accurate than (1.1).

THEOREM 2.4 Let $M \in \Re^{n\times n}$ be an H-matrix with the positive diagonal part D, let $q \in \Re^n, \hat{x} \in \Re^n_+$ be given. Let $\tilde{x}$, $\tilde{M}$ and $\tilde{q}$ be defined by (2.3), (2.5) and (2.6), respectively. Let $\bar{M}$ be the comparison matrix of M. (Note that we have $\tilde{M} = \bar{M}$ since M has the positive diagonal part.) Then we have

$$\|\tilde{x}+\tilde{M}^{-1}\max(0,-\tilde{q})\|_p \leqslant \|\bar{M}^{-1}\max(D,I)\|_p\|\min(\hat{x},M\hat{x}+q)\|_p. \tag{2.12}$$

Proof. Let $\tilde{y}$ be defined by (2.4). At first, we prove

$$\max(\tilde{M}\tilde{x},-\tilde{y}) \leqslant \max(D,I)|\min(\hat{x},M\hat{x}+q)|.$$

Consider any index $i \in \alpha$, for which we have $\hat{x}_i \leqslant (M\hat{x}+q)_i$. From (2.3) and (2.4), the definitions of $\tilde{x}$ and $\tilde{y}$, it follows that

$$(\tilde{M}\tilde{x})_i = m_{ii}\tilde{x}_i - \sum_{j\neq i}|m_{ij}|\tilde{x}_j \leqslant m_{ii}|\hat{x}_i| \leqslant \max(m_{ii},1)|\hat{x}_i|$$

and

$$-\tilde{y}_i = -(M\hat{x}+q)_i \leqslant -\hat{x}_i \leqslant \max(m_{ii},1)|\hat{x}_i|.$$

So, we have for $i \in \alpha$

$$\max((\tilde{M}\tilde{x})_i,-\tilde{y}_i) \leqslant \max(m_{ii},1)|\hat{x}_i| = \max(m_{ii},1)|\min(\hat{x}_i,(M\hat{x}+q)_i)|.$$

Consider any index $i \notin \alpha$, for which we have $\hat{x}_i > (M\hat{x}+q)_i$. From (2.3), the definition of $\tilde{x}$, it follows that $\tilde{x}_i = 0$, and so

$$(\tilde{M}\tilde{x})_i = m_{ii}\tilde{x}_i - \sum_{j\neq i}|m_{ij}|\tilde{x}_j = -\sum_{j\neq i}|m_{ij}|\tilde{x}_j \leqslant \max(m_{ii},1)|(M\hat{x}+q)_i|.$$

From (2.4), the definition of $\tilde{y}$, it follows that

$$-\tilde{y}_i = |(M\hat{x}+q)_i| \leqslant \max(m_{ii},1)|(M\hat{x}+q)_i|.$$

So, we have for $i \notin \alpha$

$$\max((\tilde{M}\tilde{x})_i,-\tilde{y}_i) \leqslant \max(m_{ii},1)|\hat{x}_i| = \max(m_{ii},1)|\min(\hat{x}_i,(M\hat{x}+q)_i)|.$$

To summarize, we have that

$$\max(\tilde{M}\tilde{x},-\tilde{y}) \leqslant \max(D,I)|\min(\hat{x},M\hat{x}+q)|.$$

Since M is an H-matrix whose diagonal elements are all positive, the matrix $\tilde{M}$ has a nonnegative inverse. Thus, it follows that

$$\tilde{M}^{-1}\max(\tilde{M}\tilde{x},-\tilde{y}) \leqslant \tilde{M}^{-1}\max(D,I)|\min(\hat{x},M\hat{x}+q)|.$$

This, together with the relation

$$\tilde{M}^{-1}\max(\tilde{M}\tilde{x},-\tilde{y}) = \tilde{M}^{-1}(\tilde{M}\tilde{x}+\max(0,-\tilde{M}\tilde{x}-\tilde{y})) = \tilde{x}+\tilde{M}^{-1}\max(0,-\tilde{q})$$

yields

$$\tilde{x}+\tilde{M}^{-1}\max(0,-\tilde{q}) \leqslant \tilde{M}^{-1}\max(D,I)|\min(\hat{x},M\hat{x}+q)|.$$

Because the norm $\|\cdot\|_p$ is monotone, (2.12) follows from the above inequality. This completes the proof. □

Finally, we show that $\varphi(\hat{x})=\varphi_H(\hat{x})=0$ if $\hat{x}$ is a solution of $LCP(M,q)$. It means that the two error bounds (2.9) and (2.11) are tight for all points in the solution set.

THEOREM 2.5 If $\hat{x}$ is a solution of $LCP(M,q)$, then we have $\varphi(\hat{x})=\varphi_H(\hat{x})=0$.

Proof. Let $\tilde{x}$ and $\tilde{q}$ be defined for $\hat{x}=x^*$ by (2.3) and (2.6), respectively. It is easy to test that $\tilde{x}=0$, $\tilde{q}\geqslant 0$. So we have $\varphi_H(\hat{x})=\tilde{x}+\tilde{M}^{-1}\max(0,-\tilde{q})=0$.

It is clear that $u^*=0$ is the least element of $FEA(\tilde{M},\tilde{q})$, where $\tilde{M}$ is defined by (2.5). Therefore, by the definition (2.8) we have $\varphi(\hat{x})=\tilde{x}+u^*=0$. This shows that the theorem is true. □

3. Numerical Experiment

In this section we perform numerical experiment

- to illustrate the error bound (2.9), and
- to show that the error bound (2.11) is more accurate than (1.1).

EXAMPLE 1 We consider the problem $LCP(M,q)$ with the following $M=(m_{ij})\in\Re^{2\times 2}$ and $q\in\Re^2$:

$$M=\begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix}, \qquad q=\begin{bmatrix} -1 \\ -1 \end{bmatrix}.$$

We choose $\hat{x}=[\frac{1}{4},\frac{5}{4}]^T$. By the definition (2.3), (2.5) and (2.6) we have

$$\tilde{x}=\begin{bmatrix} \frac{1}{4} \\ 0 \end{bmatrix}, \qquad \tilde{M}=\begin{bmatrix} 0 & -2 \\ -1 & 1 \end{bmatrix}, \qquad \tilde{q}=\begin{bmatrix} \frac{3}{2} \\ -\frac{3}{4} \end{bmatrix}.$$

By simple computation we get the least element $u^*=[0,\frac{3}{4}]^T$ of the feasible set $FEA(\tilde{M},\tilde{q})$, and conclude from COROLLARY 2.1 that $LCP(M,q)$ has a solution x^* with

$$|\hat{x}-x^*| \leqslant \varphi(\hat{x}) = \tilde{x}+u^* = \begin{bmatrix} \frac{1}{4} \\ \frac{3}{4} \end{bmatrix}.$$

Actually, $LCP(M,q)$ has two solutions $[0,1]^T$ and $[\frac{1}{2},\frac{1}{2}]^T$. The former one satisfies our componentwise error estimation with equality.

Table 1. Values of κ for *Example 2*

	$n=100$	$n=400$	$n=900$	$n=1600$	$n=2500$	$n=3600$
$\varepsilon=1e+1$	1.3463e-3	1.3063e-2	7.7890e-3	4.6964e-3	2.3229e-3	2.1577e-3
$\varepsilon=1e+0$	1.0740e-2	3.2756e-3	4.7064e-3	5.6017e-3	3.2634e-3	3.6529e-3
$\varepsilon=1e-1$	1.7994e-3	8.8600e-3	6.6397e-3	6.8918e-3	4.6962e-3	4.7909e-3
$\varepsilon=1e-2$	5.6892e-3	8.2067e-3	4.2431e-3	3.2666e-3	3.3089e-3	4.2454e-3
$\varepsilon=1e-3$	5.2584e-3	2.7136e-3	7.3654e-3	2.3826e-3	5.7360e-3	3.1293e-3
$\varepsilon=1e-4$	2.0426e-2	5.9763e-3	3.7591e-3	3.7955e-3	4.7315e-3	4.3478e-3

EXAMPLE 2 We consider the problem $LCP(M,q)$ with $M=(m_{ij})\in\Re^{n\times n}$,

$$M=\begin{bmatrix} H & -I & & \\ -I & H & \ddots & \\ & \ddots & \ddots & -I \\ & & -I & H \end{bmatrix},$$

where $H\in\Re^{k\times k}$, $k=\sqrt{n}$,

$$H=\begin{bmatrix} 4 & -1 & & \\ -1 & 4 & \ddots & \\ & \ddots & \ddots & -1 \\ & & -1 & 4 \end{bmatrix}.$$

It can be verified that M is an H-matrix. We let $x^*=(x_i^*)\in\Re^n$ with $x_i^*=\max(0,v_i-0.5)\times 10^{10(w_i-0.5)}$, $q=(q_i)\in\Re^n$ with

$$q_i=\begin{cases} -(Mx^*)_i & x_i^*>0, \\ -(Mx^*)_i+\max(0,\tilde{v}_i-0.5)\times 10^{10(\tilde{w}_i-0.5)} & x_i^*=0, \end{cases}$$

and $\hat{x}=(\hat{x}_i)\in\Re^n_+$ with $\hat{x}_i=x_i^*+\varepsilon\breve{w}_i$, where w_i, v_i, $\tilde{w}_i$, $\tilde{v}_i$ and $\breve{w}_i$ are random numbers in $[0,1]$. Such an LCP appears frequently in modeling obstacle problems, see Rodrigues (1987). It is easy to see that x^* is the solution of $LCP(M,q)$. For different choices of the dimension n and of ε we report in Table 1 the values of κ:

$$\kappa:=\frac{\|\tilde{x}+\tilde{M}^{-1}\max(0,-\tilde{q})\|_\infty}{\|\bar{M}^{-1}\max(D,I)\|_\infty\|\min(\hat{x},M\hat{x}+q)\|_\infty}.$$

The numerical results illustrate that the error bound (2.11) is much smaller than the error bound (1.1).

4. Final Remarks

In this article we present an approach of computing bound of the error $|\hat{x}-x^*|$, where x^* is a solution of $LCP(M,q)$, $\hat{x}$ is a given vector. The following are some remarks on the accuracy and the computational cost of our error bounds.

- For the case that M is an H-matrix, our error bound (2.11) was proved to be more accurate than the bound (1.1) given by Chen and Xiang. Numerical tests indicate that the new bound is much sharper.

- For the case of M being a P-matrix, that is a matrix whose all principal minors are positive, the error bound

$$\|\hat{x}-x^*\|_p \leqslant \max_{d\in[0,1]^n} \|(I-D+DM)^{-1}\|_p \|\min(\hat{x}, M\hat{x}+q)\|_p \tag{4.1}$$

was given in Chen & Xiang (2006), where $d=(d_i)\in\mathfrak{R}^n$, $d_i\in[0,1]$, $i=1,\dots,n$ and $D=diag(d)$. So far we get no theoretical results on the comparison of the accuracy of the error bounds (2.9) and (4.1).

Some numerical examples indicate that (2.9) could be more accurate than (4.1). For illustration, consider the problem $LCP(M,q)$

$$M=\begin{bmatrix} 1 & -4 \\ 5 & 7 \end{bmatrix}, \qquad q=\begin{bmatrix} -3 \\ 1 \end{bmatrix}.$$

It is easy to see that M is a P-matrix and $LCP(M,q)$ has a unique solution $x^*=[3,0]^T$. This problem was studied in Chen & Xiang (2006). We choose $\hat{x}=[4,1]^T$ and compute

$$\tilde{x}=\begin{bmatrix} 0 \\ 1 \end{bmatrix}, \qquad \tilde{M}=\begin{bmatrix} 1 & -4 \\ -5 & 7 \end{bmatrix}, \qquad \tilde{q}=\begin{bmatrix} -7 \\ 35 \end{bmatrix}.$$

It is easy to verify that $u^*=[7,0]^T$ is the least element of the feasible set $FEA(\tilde{M},\tilde{q})$. So, from (2.9) we obtain the error bound

$$|\hat{x}-x^*| \leqslant \varphi(\hat{x}) = \tilde{x}+u^* = \begin{bmatrix} 7 \\ 1 \end{bmatrix}.$$

Obviously we have $\|\hat{x}-x^*\|_1 \leqslant \|\varphi(\hat{x})\|_1 = 8$ and $\|\hat{x}-x^*\|_\infty \leqslant \|\varphi(\hat{x})\|_\infty = 7$. While the error bound (4.1) yields

$$\|\hat{x}-x^*\|_1 \leqslant 20 \quad \text{and} \quad \|\hat{x}-x^*\|_\infty \leqslant 15.$$

See Chen & Xiang (2006). This indicates that for this special example the bound (2.9) is more accurate than (4.1).

- Our error bounds can be computed if an element of the feasible set $FEA(\tilde{M},\tilde{q})$ is available, and such an element can be obtained by solving a convex quadratic programming (2.10) in general case, and by solving a linear system in the case of H-matrix. The exact solution of the convex programming (2.10) gives a "sharpest" error bound (2.9), while any feasible solution of (2.10) delivers an error bound (2.7). For the latter case, one can expect a smaller computational cost.

- The problem when the feasible set $FEA(\tilde{M},\tilde{q})$ is empty, remains open. However, whenever $FEA(\tilde{M},\tilde{q}) \neq \emptyset$, our error estimation can work efficiently. It is due to the convex programming formulation (2.10).
- At last we mention that our error estimation may work in a loose setting, say for the problem $LCP(M,q)$ studied in EXAMPLE 1, where M is not a P-matrix. Actually M is even not an R_0-matrix. M is called an R_0-matrix if $LCP(M,0)$ has a unique solution. It is well known that a P-matrix is an R_0-matrix. Error bound for $LCP(M,q)$, where M is not an R_0-matrix, has not been studied yet.

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