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Componentwise error bounds for linear complementarity problems

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Componentwise error bounds for linear complementarity problems are presented. For the problem with an H-matrix the error bound can be computed by solving a system of linear equations. It is proved that our error bound is more accurate than that obtained recently by Chen & Xiang (2006, *Math. Prog., Ser. A*, **106**, 513–525). Numerical results show that the new bound is often much better than previous ones.

Keywords: error bound; linear complementarity problem; convex quadratic programming.

1. Introduction

Let $M \in \Re^{n\times n}$ and $q \in \Re^n$ be given. The linear complementarity problem (LCP), denoted by LCP (M, q), is to find a vector $x^* \in \Re^n$ such that

$$x^* \geqslant 0, \quad Mx^* + q \geqslant 0, \quad (x^*)^{\mathrm{T}}(Mx^* + q) = 0,$$

or to show that no such x^* exists. Here the inequalities are meant componentwise. LCPs have many important real-world applications, for example, see Cottle *et al.* (1992) and Ferris & Pang (1997).

Let $\hat{x} \in \Re^n_+$ be an arbitrary but fixed vector. Estimation of the error $\hat{x} - x^*$ plays an important role in both the numerical solution and theoretical analysis. Norm error estimation for LCPs has been extensively studied so far, for example, see Chen & Xiang (2006), Mangasarian & Ren (1994), Mathias & Pang (1990) and Pang (1997).

In this paper we present new componentwise error bounds, that is, we give $r \in \Re^n_+$ such that

$$|\hat{x} - x^*| \leqslant r,$$

where $|y|$ means the vector whose ith component is $|y_i|$. Such an r can be computed in general by finding a feasible solution of a convex quadratic programming problem for which a very mild computational

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cost is expected. The solution of the convex quadratic programming problem delivers a tight error bound.

For the case where M is an H-matrix, that is, the comparison matrix $\bar{M} = (\bar{m}_{ij}) \in \Re^{n\times n}$ has a non-negative inverse, where

$$\bar{m}_{ij} = \begin{cases} |m_{ii}| & \text{if } i = j, \\ -|m_{ij}| & \text{if } i \neq j, \end{cases}$$

we present a tight error bound $\varphi_H(\hat{x})$ that can be computed by solving a system of linear equations. For an LCP with an H-matrix the error bound

$$\|\hat{x} - x^*\|_p \leqslant \|\bar{M}^{-1} \max(D, I)\|_p \| \min(\hat{x}, M\hat{x} + q)\|_p =: \phi_p(\hat{x}) \tag{1.1}$$

was given by Chen & Xiang (2006), where $p \in [1, +\infty]$, D is the diagonal part of M, I is the identity matrix, and 'max' and 'min' are taken componentwise. The error bound $\phi_p(\hat{x})$ was proved to be more accurate than the well-known bound given by Mathias & Pang (1990). We will prove that our error bound $\varphi_H(\hat{x})$ is more accurate than the error bound $\phi_p(\hat{x})$ given in (1.1) in the sense that

$$\|\varphi_H(\hat{x})\|_p \leqslant \phi_p(\hat{x}).$$

The numerical results illustrate that this inequality holds strictly and for different orders of magnitude for our model problem.

To conclude this section we give some notation. The set

$$\text{FEA}(M, q) := \{x \in \Re^n\colon\ x \geqslant 0,\ Mx + q \geqslant 0\} \tag{1.2}$$

is called the *feasible set* of the problem LCP(M, q). An element of FEA(M, q) is called a *feasible vector* of the problem LCP(M, q). This feasible vector is also known as a *feasible solution*.

For $x = (x_i)$ and $y = (y_i) \in \Re^n$ we have that $x \leqslant y$ stands for $x_i \leqslant y_i$, where $i = 1, \ldots, n$. We denote by $\max(x, y)$ and by $\min(x, y)$ the componentwise maximum and minimum of x and y, respectively. Let $\underline{a} = (\underline{a}_i) \in \Re^n$ and $\overline{a} = (\overline{a}_i) \in \Re^n$ be given with $\underline{a} \leqslant \overline{a}$. We define an n-dimensional interval vector as the set of vectors

$$[a] = [\underline{a}, \overline{a}] := \{x \in \Re^n\colon \underline{a} \leqslant x \leqslant \overline{a}\}.$$

For an interval vector $[a] = [\underline{a}, \overline{a}]$ we define the operation

$$\max(0, [a]) := [\max(0, \underline{a}), \max(0, \overline{a})].$$

For simplicity, we write $[a, a] = a$ for $a \in \Re^n$. The operations $+$, $-$ and $\times$ can be defined for intervals. We refer to Neumaier (1990) for details.

2. Error bound

We begin our study with an existence theorem. It gives a sufficient condition for guaranteeing that an interval vector contains a solution of an LCP.

THEOREM 2.1 Let $M \in \Re^{n\times n}$ and $q \in \Re^n$ be given. Let $[x]$ be an n-dimensional interval vector and let $\hat{x} \in [x]$ be an arbitrary but fixed vector. Let a diagonal matrix $\varDelta = \text{diag}(\delta_i) \in \Re^{n\times n}$ be given with

$\delta_i > 0$, where $i = 1, \ldots, n$. If

$$\Gamma(\hat{x}, [x], \Delta) := \max\left(0, \hat{x} - \Delta(M\hat{x} + q) + (I - \Delta M)([x] - \hat{x})\right) \subseteq [x] \tag{2.1}$$

then LCP(M, q) has a solution $x^* \in \Gamma(\hat{x}, [x], \Delta)$.

For the proof of Theorem 2.1 we refer to Alefeld *et al.* (2004).

Now we find a vector $r \in \Re^n_+$ such that the condition (2.1) holds for the interval vector $[\hat{x} - r, \hat{x} + r]$. Consequently, it follows from Theorem 2.1 that LCP(M, q) has a solution $x^* \in [\hat{x} - r, \hat{x} + r]$, which implies the componentwise error bound

$$|\hat{x} - x^*| \leqslant r.$$

Given a vector $\hat{x} \in \Re^n_+$, we define

$$\alpha := \{i\colon\ \hat{x}_i \leqslant (M\hat{x} + q)_i\}, \tag{2.2}$$

$$\tilde{x} := (\tilde{x}_i) \quad \text{with } \tilde{x}_i = \begin{cases} \hat{x}_i & \text{if } i \in \alpha, \\ 0 & \text{if } i \notin \alpha, \end{cases} \tag{2.3}$$

$$\tilde{y} := (\tilde{y}_i) \quad \text{with } \tilde{y}_i = \begin{cases} (M\hat{x} + q)_i & \text{if } i \in \alpha, \\ -|(M\hat{x} + q)_i| & \text{if } i \notin \alpha, \end{cases} \tag{2.4}$$

$$\tilde{M} := D - |B|, \tag{2.5}$$

$$\tilde{q} := \tilde{M}\tilde{x} + \tilde{y}, \tag{2.6}$$

where D and $-B$ are the diagonal and the off-diagonal parts of M, respectively.

Using these definitions, we can give our main error estimation result as follows.

THEOREM 2.2 Let $M \in \Re^{n\times n}$, $q \in \Re^n$ and $\hat{x} \in \Re^n_+$ be given. Let $\tilde{M} \in \Re^{n\times n}$ and $\tilde{q} \in \Re^n$ be defined by (2.5) and (2.6), respectively. If $u \in$ FEA$(\tilde{M}, \tilde{q})$ then LCP(M, q) has a solution $x^* \in [x] := [\hat{x} - r, \hat{x} + r]$, where $r = \tilde{x} + u$ and $\tilde{x} \in \Re^n$ is defined by (2.3). As a direct consequence, we have the error bound

$$|\hat{x} - x^*| \leqslant r. \tag{2.7}$$

Proof. It is sufficient to show that condition (2.1) holds for $r = \tilde{x} + u$. We define a positive-definite diagonal matrix $\Delta = \text{diag}(\delta_i) \in \Re^{n\times n}$ by setting

$$\delta_i = \begin{cases} 1 & \text{if } m_{ii} \leqslant 0, \\ 1/m_{ii} & \text{if } m_{ii} > 0. \end{cases}$$

It can be verified that (see Neumaier, 1990)

$$(I - \Delta M)([x] - \hat{x}) = [-|I - \Delta M|r, |I - \Delta M|r].$$

By the definition of $\Gamma(\hat{x}, [x], \Delta)$, we have $\Gamma(\hat{x}, [x], \Delta) = [\underline{\Gamma(\hat{x}, [x], \Delta)}, \overline{\Gamma(\hat{x}, [x], \Delta)}]$, where

$$\underline{\Gamma(\hat{x}, [x], \Delta)} = \max(0, \hat{x} - \Delta(M\hat{x} + q) - |I - \Delta M|r),$$

$$\overline{\Gamma(\hat{x}, [x], \Delta)} = \max(0, \hat{x} - \Delta(M\hat{x} + q) + |I - \Delta M|r).$$

First, we prove that $\overline{\Gamma(\hat{x}, [x], \Delta)} \leqslant \hat{x} + r$. Remembering that D and $-B$ are the diagonal and off-diagonal parts of M, respectively, we have

$$|I - \Delta M| r = r - \Delta(D - |B|) r = r - \Delta \tilde{M} r.$$

The definition (2.4) yields $M\hat{x} + q \geqslant \tilde{y}$. Note that $\tilde{M}\tilde{x} + \tilde{y} = \tilde{q}$, $r = \tilde{x} + u$ and $\tilde{M}u + \tilde{q} \geqslant 0$. Thus it follows that

$$\begin{aligned} \hat{x} - \Delta(M\hat{x} + q) + |I - \Delta M| r &\leqslant \hat{x} - \Delta\tilde{y} + r - \Delta\tilde{M}r \\ &\leqslant \hat{x} + r - \Delta\tilde{y} - \Delta\tilde{M}(\tilde{x} + u) \\ &= \hat{x} + r - \Delta(\tilde{M}u + \tilde{M}\tilde{x} + \tilde{y}) \\ &= \hat{x} + r - \Delta(\tilde{M}u + \tilde{q}) \\ &\leqslant \hat{x} + r, \end{aligned}$$

which, together with $\hat{x} + r = \hat{x} + \tilde{x} + u \geqslant u \geqslant 0$, yields $\overline{\Gamma(\hat{x}, [x], \Delta)} \leqslant \hat{x} + r$.

Now we prove that $\underline{\Gamma(\hat{x}, [x], \Delta)} \geqslant \hat{x} - r$. For any index $i \in \alpha$ we have

$$[\underline{\Gamma(\hat{x}, [x], \Delta)}]_i = 0 \geqslant -u_i = \hat{x}_i - r_i.$$

For any index $i \notin \alpha$ we note from the definition (2.4) that $-(M\hat{x} + q)_i \geqslant \tilde{y}_i$. Considering that $\tilde{M}\tilde{x} + \tilde{y} = \tilde{q}$ and $u \in \mathrm{FEA}(\tilde{M}, \tilde{q})$ and that $r = \tilde{x} + u$, we have

$$\begin{aligned} [\underline{\Gamma(\hat{x}, [x], \Delta)}]_i &\geqslant [\hat{x} - \Delta(M\hat{x} + q) - |I - \Delta M| r]_i \\ &\geqslant [\hat{x} + \Delta\tilde{y} - r + \Delta\tilde{M}r]_i \\ &\geqslant \hat{x}_i - r_i + [\Delta\tilde{y} + \Delta\tilde{M}(\tilde{x} + u)]_i \\ &= \hat{x}_i - r_i + [\Delta(\tilde{M}u + \tilde{M}\tilde{x} + \tilde{y})]_i \\ &= \hat{x}_i - r_i + [\Delta(\tilde{M}u + \tilde{q})]_i \\ &\geqslant \hat{x}_i - r_i. \end{aligned}$$

Hence it is shown that condition (2.1) holds for $[x] := [\hat{x} - r, \hat{x} + r]$ with $r = \tilde{x} + u$. Therefore it follows from Theorem 2.1 that $\mathrm{LCP}(M, q)$ has a solution $x^* \in [x] := [\hat{x} - r, \hat{x} + r]$, which in turns implies that the error bound (2.7) holds. This completes our proof. □

We note that the matrix $\tilde{M}$, defined by (2.5), is a Z-matrix. A Z-matrix is a matrix whose all off-diagonal entries are nonpositive. So, if $\mathrm{FEA}(\tilde{M}, \tilde{q}) \neq \emptyset$, then there is a unique vector $u^* \in \mathrm{FEA}(\tilde{M}, \tilde{q})$ that is a solution of $\mathrm{LCP}(\tilde{M}, \tilde{q})$ such that, for any $u \in \mathrm{FEA}(\tilde{M}, \tilde{q})$, we have $u^* \leqslant u$ (Cottle *et al.*, 1992, pp. 198–212). This vector u^* is usually called the *least element* of the feasible set $\mathrm{FEA}(\tilde{M}, \tilde{q})$. As u^* is also a feasible vector, we obtain the following error estimation result, which is a special case of Theorem 2.2.

COROLLARY 2.3 In the setting of Theorem 2.2 we let u^* be the least element of $\mathrm{FEA}(\tilde{M}, \tilde{q})$, which is unique. We define

$$\varphi(\hat{x}) := \tilde{x} + u^*. \tag{2.8}$$

Then LCP(M, q) has a solution $x^* \in [x] := [\hat{x} - \varphi(\hat{x}), \hat{x} + \varphi(\hat{x})]$. As a direct consequence, we have the error estimate

$$|\hat{x} - x^*| \leqslant \varphi(\hat{x}). \tag{2.9}$$

Corollary 2.3 indicates that the error bound $\varphi(\hat{x})$, defined by the least element, is the 'sharpest' compared with the one defined only by a certain feasible vector of LCP$(\tilde{M}, \tilde{q})$. To compute $\varphi(\hat{x})$ we need to find the least element u^* of the feasible set FEA$(\tilde{M}, \tilde{q})$. It can be cast into the following convex quadratic programming problem:

$$\begin{aligned} &\min\ u^T u, \\ &\text{such that } \tilde{M}u + \tilde{q} \geqslant 0, \\ &\qquad u \geqslant 0, \end{aligned} \tag{2.10}$$

which can be solved efficiently using existing software, for example, using CVX by Grant *et al.* (2008). Of course, we sometimes do not need to solve the problem (2.10) exactly because any feasible solution of (2.10) provides an error bound (2.7). In this way, a much smaller computational cost can be expected.

Now we consider the special case that M is an H-matrix whose diagonal elements are all positive. An H-matrix is a matrix whose comparison matrix is an M-matrix, while an M-matrix is a matrix whose off-diagonal elements are all nonpositive and whose inverse has no negative elements (see Plemmons, 1977). In this case the problem LCP(M, q) has a unique solution for any $q \in \Re^n$ (see Cottle *et al.*, 1992, pp. 148–152). An LCP with an H-matrix appears frequently in modelling real-world problems (see, e.g., Rodrigues, 1987 and Ferris & Pang, 1997). The following theorem shows that, for such an LCP, an error bound given by (2.7) can be obtained by solving a system of linear equations.

THEOREM 2.4 Suppose that M is an H-matrix whose diagonal elements are all positive. Let $\tilde{x}$, $\tilde{M}$ and $\tilde{q}$ be defined by (2.3), (2.5) and (2.6), respectively. Then we have the estimate

$$|\hat{x} - x^*| \leqslant \varphi_H(\hat{x}) := \tilde{x} + \tilde{M}^{-1} \max(0, -\tilde{q}), \tag{2.11}$$

where x^* is the unique solution of LCP(M, q).

REMARK 2.5 The estimate (2.11) can be computed by solving a system of linear equations.

Proof of Theorem 2.4. Since M is assumed to be an H-matrix whose all diagonal elements are positive, we have $\tilde{M}^{-1} \geqslant 0$, that is, each element of $\tilde{M}^{-1}$ is non-negative. So

$$u = \tilde{M}^{-1} \max(0, -\tilde{q}) \geqslant 0,$$

$$\tilde{M}u + \tilde{q} = \tilde{M}\tilde{M}^{-1} \max(0, -\tilde{q}) + \tilde{q} = \max(0, -\tilde{q}) + \tilde{q} \geqslant 0.$$

This means that $u = \bar{M}^{-1} \max(0, -\tilde{q}) \in$ FEA$(\tilde{q}, \tilde{M})$. Therefore (2.11) follows from (2.7) in Theorem 2.2. □

We mentioned in Section 1 that the error bound (1.1) was given by Chen & Xiang (2006) for the LCP with an H-matrix. This bound was proved to be more accurate than the well-known error bound given by Mathias & Pang (1990). We now show that our error bound (2.11) is more accurate than (1.1).

THEOREM 2.6 Let $M \in \Re^{n\times n}$ be an H-matrix with positive diagonal part D, and let $q \in \Re^n$ and $\hat{x} \in \Re^n_+$ be given. Let $\tilde{x}$, $\tilde{M}$ and $\tilde{q}$ be defined by (2.3), (2.5) and (2.6), respectively. Let $\bar{M}$ be the comparison matrix of M. (Note that we have $\tilde{M} = \bar{M}$ since M has the positive diagonal part.) Then we have

$$\|\tilde{x} + \tilde{M}^{-1} \max(0, -\tilde{q})\|_p \leqslant \|\bar{M}^{-1} \max(D, I)\|_p \| \min(\hat{x}, M\hat{x} + q)\|_p. \tag{2.12}$$

Proof. Let $\tilde{y}$ be defined by (2.4). We first prove that

$$\max(\tilde{M}\tilde{x}, -\tilde{y}) \leqslant \max(D, I)|\min(\hat{x}, M\hat{x}+q)|.$$

Consider any index $i \in \alpha$ for which we have $\hat{x}_i \leqslant (M\hat{x}+q)_i$. From (2.3) and (2.4), the definitions of $\tilde{x}$ and $\tilde{y}$, it follows that

$$(\tilde{M}\tilde{x})_i = m_{ii}\tilde{x}_i - \sum_{j\neq i}|m_{ij}|\tilde{x}_j \leqslant m_{ii}|\hat{x}_i| \leqslant \max(m_{ii},1)|\hat{x}_i|$$

and

$$-\tilde{y}_i = -(M\hat{x}+q)_i \leqslant -\hat{x}_i \leqslant \max(m_{ii},1)|\hat{x}_i|.$$

So we have for $i \in \alpha$ that

$$\max((\tilde{M}\tilde{x})_i, -\tilde{y}_i) \leqslant \max(m_{ii},1)|\hat{x}_i| = \max(m_{ii},1)|\min(\hat{x}_i, (M\hat{x}+q)_i)|.$$

Consider any index $i \notin \alpha$ for which we have $\hat{x}_i > (M\hat{x}+q)_i$. From (2.3), the definition of $\tilde{x}$, it follows that $\tilde{x}_i = 0$, and so

$$(\tilde{M}\tilde{x})_i = m_{ii}\tilde{x}_i - \sum_{j\neq i}|m_{ij}|\tilde{x}_j = -\sum_{j\neq i}|m_{ij}|\tilde{x}_j \leqslant \max(m_{ii},1)|(M\hat{x}+q)_i|.$$

From (2.4), the definition of $\tilde{y}$, it follows that

$$-\tilde{y}_i = |(M\hat{x}+q)_i| \leqslant \max(m_{ii},1)|(M\hat{x}+q)_i|.$$

So we have for $i \notin \alpha$ that

$$\max((\tilde{M}\tilde{x})_i, -\tilde{y}_i) \leqslant \max(m_{ii},1)|\hat{x}_i| = \max(m_{ii},1)|\min(\hat{x}_i, (M\hat{x}+q)_i)|.$$

To summarize, we have that

$$\max(\tilde{M}\tilde{x}, -\tilde{y}) \leqslant \max(D, I)|\min(\hat{x}, M\hat{x}+q)|.$$

Since M is an H-matrix whose diagonal elements are all positive, the matrix $\tilde{M}$ has a non-negative inverse. Thus it follows that

$$\tilde{M}^{-1}\max(\tilde{M}\tilde{x}, -\tilde{y}) \leqslant \tilde{M}^{-1}\max(D, I)|\min(\hat{x}, M\hat{x}+q)|.$$

This, together with the relation

$$\tilde{M}^{-1}\max(\tilde{M}\tilde{x}, -\tilde{y}) = \tilde{M}^{-1}(\tilde{M}\tilde{x} + \max(0, -\tilde{M}\tilde{x}-\tilde{y})) = \tilde{x} + \tilde{M}^{-1}\max(0, -\tilde{q}),$$

yields

$$\tilde{x} + \tilde{M}^{-1}\max(0, -\tilde{q}) \leqslant \tilde{M}^{-1}\max(D, I)|\min(\hat{x}, M\hat{x}+q)|.$$

Because the norm $\|\cdot\|_p$ is monotone, inequality (2.12) follows from the above inequality. This completes the proof. □

Finally, we show that $\varphi(\hat{x}) = \varphi_H(\hat{x}) = 0$ if $\hat{x}$ is a solution of LCP(M, q). This means that the two error bounds (2.9) and (2.11) are tight for all points in the solution set.

THEOREM 2.7 If $\hat{x}$ is a solution of LCP(M, q) then we have $\varphi(\hat{x}) = \varphi_H(\hat{x}) = 0$.

Proof. Let $\tilde{x}$ and $\tilde{q}$ be defined for $\hat{x} = x^*$ by (2.3) and (2.6), respectively. It is easy to show that $\tilde{x} = 0$ and $\tilde{q} \geqslant 0$. So we have $\varphi_H(\hat{x}) = \tilde{x} + \tilde{M}^{-1}\max(0, -\tilde{q}) = 0$.

It is clear that $u^* = 0$ is the least element of FEA$(\tilde{M}, \tilde{q})$, where $\tilde{M}$ is defined by (2.5). Therefore, by the definition (2.8), we have $\varphi(\hat{x}) = \tilde{x} + u^* = 0$. This shows that the theorem holds. □

3. Numerical experiment

In this section we perform numerical experiments to illustrate the error bound (2.9) and to show that the error bound (2.11) is more accurate than (1.1).

EXAMPLE 3.1 We consider the problem LCP(M, q) with the following $M = (m_{ij}) \in \Re^{2\times 2}$ and $q \in \Re^2$:

$$M = \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix}, \quad q = \begin{bmatrix} -1 \\ -1 \end{bmatrix}.$$

We choose $\hat{x} = \left[\frac{1}{4}, \frac{5}{4}\right]^{\mathrm{T}}$. By the definitions (2.3), (2.5) and (2.6), we have

$$\tilde{x} = \begin{bmatrix} \frac{1}{4} \\ 0 \end{bmatrix}, \quad \tilde{M} = \begin{bmatrix} 0 & -2 \\ -1 & 1 \end{bmatrix}, \quad \tilde{q} = \begin{bmatrix} \frac{3}{2} \\ -\frac{3}{4} \end{bmatrix}.$$

By a simple computation, we obtain the least element $u^* = \left[0, \frac{3}{4}\right]^{\mathrm{T}}$ of the feasible set FEA$(\tilde{M}, \tilde{q})$ and conclude from Corollary 2.3 that LCP(M, q) has a solution x^* with

$$|\hat{x} - x^*| \leqslant \varphi(\hat{x}) = \tilde{x} + u^* = \begin{bmatrix} \frac{1}{4} \\ \frac{3}{4} \end{bmatrix}.$$

Actually, LCP(M, q) has two solutions $[0, 1]^{\mathrm{T}}$ and $\left[\frac{1}{2}, \frac{1}{2}\right]^{\mathrm{T}}$. The former one satisfies our componentwise error estimation with equality.

EXAMPLE 3.2 We consider the problem LCP(M, q) with $M = (m_{ij}) \in \Re^{n\times n}$ and

$$M = \begin{bmatrix} H & -I & & \\ -I & H & \ddots & \\ & \ddots & \ddots & -I \\ & & -I & H \end{bmatrix},$$

where $H \in \Re^{k\times k}$, $k = \sqrt{n}$ and

$$H = \begin{bmatrix} 4 & -1 & & \\ -1 & 4 & \ddots & \\ & \ddots & \ddots & -1 \\ & & -1 & 4 \end{bmatrix}.$$

TABLE 1 *Values of κ for Example 3.2*

ϵ	$n = 100$	$n = 400$	$n = 900$	$n = 1600$	$n = 2500$	$n = 3600$
10^1	1.3463×10^{-3}	1.3063×10^{-2}	7.7890×10^{-3}	4.6964×10^{-3}	2.3229×10^{-3}	2.1577×10^{-3}
10^0	1.0740×10^{-2}	3.2756×10^{-3}	4.7064×10^{-3}	5.6017×10^{-3}	3.2634×10^{-3}	3.6529×10^{-3}
10^{-1}	1.7994×10^{-3}	8.8600×10^{-3}	6.6397×10^{-3}	6.8918×10^{-3}	4.6962×10^{-3}	4.7909×10^{-3}
10^{-2}	5.6892×10^{-3}	8.2067×10^{-3}	4.2431×10^{-3}	3.2666×10^{-3}	3.3089×10^{-3}	4.2454×10^{-3}
10^{-3}	5.2584×10^{-3}	2.7136×10^{-3}	7.3654×10^{-3}	2.3826×10^{-3}	5.7360×10^{-3}	3.1293×10^{-3}
10^{-4}	2.0426×10^{-2}	5.9763×10^{-3}	3.7591×10^{-3}	3.7955×10^{-3}	4.7315×10^{-3}	4.3478×10^{-3}

It can be verified that M is an H-matrix. We let $x^* = (x_i^*) \in \Re^n$ with $x_i^* = \max(0, v_i - 0.5) \times 10^{10(w_i - 0.5)}$, $q = (q_i) \in \Re^n$ with

$$q_i = \begin{cases} -(Mx^*)_i & x_i^* > 0, \\ -(Mx^*)_i + \max(0, \tilde{v}_i - 0.5) \times 10^{10(\tilde{w}_i - 0.5)} & x_i^* = 0, \end{cases}$$

and $\hat{x} = (\hat{x}_i) \in \Re^n_+$ with $\hat{x}_i = x_i^* + \epsilon \breve{w}_i$, where $w_i, v_i, \tilde{w}_i, \tilde{v}_i$ and $\breve{w}_i$ are random numbers in [0, 1]. Such an LCP appears frequently in modelling obstacle problems (see Rodrigues, 1987). It is easy to see that x^* is the solution of LCP(M, q). For different choices of the dimension n and of ϵ we report in Table 1 the values of

$$\kappa := \frac{\|\tilde{x} + \tilde{M}^{-1} \max(0, -\tilde{q})\|_\infty}{\|\bar{M}^{-1} \max(D, I)\|_\infty \| \min(\hat{x}, M\hat{x} + q)\|_\infty}.$$

The numerical results illustrate that the error bound (2.11) is much smaller than the error bound (1.1).

4. Final remarks

In this article we have presented an approach for computing the bound of the error $|\hat{x} - x^*|$, where x^* is a solution of LCP(M, q) and $\hat{x}$ is a given vector. The following are some remarks on the accuracy and the computational cost of our error bounds.

- For the case that M is an H-matrix our error bound (2.11) was proved to be more accurate than the bound (1.1) given by Chen and Xiang. Numerical tests indicated that the new bound is much sharper.
- For the case of M being a P-matrix, that is, a matrix whose principal minors are all positive, the error bound

$$\|\hat{x} - x^*\|_p \leqslant \max_{d \in [0,1]^n} \|(I - D + DM)^{-1}\|_p \| \min(\hat{x}, M\hat{x} + q)\|_p \tag{4.1}$$

was given in Chen & Xiang (2006), where $d = (d_i) \in \Re^n$, $d_i \in [0, 1]$ for $i = 1, \ldots, n$ and $D = \operatorname{diag}(d)$. So far, no theoretical results have been obtained that compare the accuracy of the error bounds (2.9) and (4.1).
Some numerical examples indicated that (2.9) could be more accurate than (4.1). For an illustration we consider the problem LCP(M, q) with

$$M = \begin{bmatrix} 1 & -4 \\ 5 & 7 \end{bmatrix}, \quad q = \begin{bmatrix} -3 \\ 1 \end{bmatrix}.$$

It is easy to see that M is a P-matrix and LCP(M, q) has a unique solution $x^* = [3, 0]^{\mathrm{T}}$. This problem was studied in Chen & Xiang (2006). We choose $\hat{x} = [4, 1]^{\mathrm{T}}$ and compute

$$\tilde{x} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \tilde{M} = \begin{bmatrix} 1 & -4 \\ -5 & 7 \end{bmatrix}, \quad \tilde{q} = \begin{bmatrix} -7 \\ 35 \end{bmatrix}.$$

It is easy to verify that $u^* = [7, 0]^{\mathrm{T}}$ is the least element of the feasible set FEA$(\tilde{M}, \tilde{q})$. So from (2.9) we obtain the error bound

$$|\hat{x} - x^*| \leqslant \varphi(\hat{x}) = \tilde{x} + u^* = \begin{bmatrix} 7 \\ 1 \end{bmatrix}.$$

Obviously, we have $\|\hat{x} - x^*\|_1 \leqslant \|\varphi(\hat{x})\|_1 = 8$ and $\|\hat{x} - x^*\|_\infty \leqslant \|\varphi(\hat{x})\|_\infty = 7$. While the error bound (4.1) yields

$$\|\hat{x} - x^*\|_1 \leqslant 20 \quad \text{and} \quad \|\hat{x} - x^*\|_\infty \leqslant 15.$$

(see Chen & Xiang, 2006). This indicates that, for this special example, the bound (2.9) is more accurate than (4.1).

- Our error bounds can be computed if an element of the feasible set FEA$(\tilde{M}, \tilde{q})$ is available, and such an element can be obtained by solving a convex quadratic programming problem (2.10) in the general case, and by solving a linear system in the case of an H-matrix. The exact solution of the convex programming problem (2.10) gives a ‘sharpest’ error bound (2.9), while any feasible solution of (2.10) delivers an error bound (2.7). For the latter case one can expect a smaller computational cost.
- The problem remains open when the feasible set FEA$(\tilde{M}, \tilde{q})$ is empty. Otherwise, our algorithm works efficiently, due to the convex programming formulation (2.10).
- Finally, we mention that our error estimation may work in a loose setting, say for the problem LCP(M, q) studied in Example 1, where M is not a P-matrix. Actually, M is not even an R_0-matrix. M is called an R_0-matrix if LCP$(M, 0)$ has a unique solution. It is well known that a P-matrix is an R_0-matrix. The error bound for LCP(M, q), where M is not an R_0-matrix, has not yet been studied.

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