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AN ONLY 2-STEP Q-SUPERLINEAR CONVERGENCE EXAMPLE FOR SOME ALGORITHMS THAT USE REDUCED HESSIAN APPROXIMATIONS

Y. YUAN

Department of Applied Mathematics and Theoretical Physics, University of Cambridge, Cambridge, England

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It is shown by example that the reduced Hessian method for constrained optimization that is known to give 2-step Q-superlinear convergence may not converge Q-superlinearly.

Key words: Constrained Optimization, Reduced Hessian, Q-superlinear Convergence.

1. Introduction

Consider the following constrained optimization problem

$$\begin{aligned} &\text{minimize} \quad f(x) \\ &\text{subject to} \quad c(x)=0 \end{aligned} \tag{1.1}$$

where $f(x)$ from $\mathbb{R}^n$ to $\mathbb{R}$ and $c(x)$ from $\mathbb{R}^n$ to $\mathbb{R}^m$ are twice continuously differentiable functions. We define $A(x)=\nabla c^{\mathrm{T}}(x)$ and $g(x)=\nabla f(x)$. Assume that x^* is a solution of (1.1), that $A(x^*)$ has full column rank, and that $\lambda^*\in\mathbb{R}^m$ is the unique Lagrangian multiplier vector that satisfies

$$g(x^*)-A(x^*)\lambda^*=0. \tag{1.2}$$

Let the Q-R factorization of $A(x)$ be

$$[Y(x), Z(x)]\begin{bmatrix} R(x) \\ 0 \end{bmatrix}, \tag{1.3}$$

and assume that $Y(x)$, $Z(x)$ and $R(x)$ are continuously differentiable. Further, we assume that the second order sufficiency condition holds at x^*, that is $Z^{\mathrm{T}}(x^*)W^*Z(x^*)$ is positive definite, where

$$W^*=\nabla^2 f(x^*)-\sum_{i=1}^{m}\lambda_i^*\nabla^2 c_i(x^*), \tag{1.4}$$

$\lambda_i^*(i=1,2,\ldots,m)$ being the components of λ^*.

The two-sided reduced Hessian approximation method generates $\{x_k\}$ in the following way: given x_k and an $(n-m)\times(n-m)$ matrix B_k, which is the reduced

Hessian approximation, let p_k be a solution of the square linear system

$$\begin{bmatrix} B_k Z^{\mathrm{T}}(x_k) \\ A^T(x_k) \end{bmatrix} p_k = -\begin{bmatrix} Z^{\mathrm{T}}(x_k) g(x_k) \\ c(x_k) \end{bmatrix}, \tag{1.5}$$

and set

$$x_{k+1} = x_k + p_k. \tag{1.6}$$

For more details of the method, see Nocedal and Overton [3].

The method is 2-step Q-superlinearly convergent, due to the following result (see Powell [5], Nocedal and Overton [3]):

Theorem. *If $x_k \to x^*$, if $\|B_k^{-1}\|$ is uniformly bounded, and if*

$$\frac{\|[B_k - Z^{\mathrm{T}}(x^*) W^* Z(x^*)] Z^{\mathrm{T}}(x_k)(x_{k+1} - x_k)\|}{\|x_{k+1} - x_k\|} \to 0,$$

then $x_k \to x^*$ at a 2-step Q-superlinear rate in the sense that

$$\|x_{k+1} - x^*\| / \|x_{k-1} - x^*\| \to 0. \tag{1.7}$$

The main purpose of this paper is to consider the following question. If $B_k \to Z^{\mathrm{T}}(x^*) W^* Z(x^*)$, which gives the limit (1.7), is it possible that the rate of convergence of alternate iterations is no better than linear? The example that is presented in the next section shows that this possiblity can occur. To make our analysis simple we let $B_k = Z^{\mathrm{T}}(x^*) W^* Z(x^*)$ for all k.

2. The example

Let $n = 2$, $m = 1$. We consider an example which satisfies

$$g(x^*) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \qquad A(x^*) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad \lambda^* = 0, \tag{2.1}$$

where $x^* = (0, 0)^{\mathrm{T}}$ is a solution of (1.1) at which the second order sufficiency condition holds. Let the Q–R factors of $A(x^*)$ be the matrices

$$Y(x^*) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad Z(x^*) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \qquad R^* = 1. \tag{2.2}$$

Further, $f(x)$ will be chosen such that

$$Z^{\mathrm{T}}(x^*) W^* Z(x^*) = 1. \tag{2.3}$$

We let $B_k \equiv 1$ for all k, as mentioned at the end of section 1. It will be shown that $\{x_k\}$ generated by the method given in section 1 may converge to x^* by oscillating between the two curves $y = z$ and $y = z^2$, where y and z are the components of x, i.e. $x=(y, z)^{\mathrm{T}}$. Specifically we obtain $x_{2k}=(t_k, t_k)^{\mathrm{T}}$ and $x_{2k+1}=(t_k^2, t_k)^{\mathrm{T}}$ for all k, where $t_{k+1}=t_k^2$ and $|t_1|$ is very small. By induction, this statement can be proved by showing that, for $x_k=(y_k, z_k)^{\mathrm{T}}$, the system (1.5) has the solution

$$p_k=\begin{bmatrix} z_k^2-y_k \\ y_k-z_k \end{bmatrix}, \tag{2.4}$$

if $\|x_k\|_2$ is very small and if x_k satisfies either $y_k=z_k$ or $y_k=z_k^2$.

As the Jacobian of the constraint is

$$A(x)=\begin{bmatrix} \dfrac{\partial c(y, z)}{\partial y} \\ \dfrac{\partial c(y, z)}{\partial z} \end{bmatrix} \tag{2.5}$$

and the Q-R factorization of $A(x)$ is (1.3), we have

$$Y(x)=\frac{1}{\|A(x)\|_2}A(x),$$

$$Z(x)=\frac{1}{\|A(x)\|_2}\begin{bmatrix} \dfrac{-\partial c(y, z)}{\partial z} \\ \dfrac{\partial c(y, z)}{\partial y} \end{bmatrix}, \tag{2.6}$$

$$R(x)=\|A(x)\|_2.$$

We require $\partial c(y, z)/\partial y$ and $\partial c(y, z)/\partial z$ to be continuously differentiable functions such that $\partial c(0, 0)/\partial y=1$ and $\partial c(0, 0)/\partial z=0$, $\|\cdot\|_2$ being the Euclidean norm.

To construct $f(x)$ and $c(x)$ such that (2.4) holds for all k, remembering that $B_k=1$, it is sufficient to obtain

$$Z^{\mathrm{T}}(x)\begin{bmatrix} z^2-y \\ y-z \end{bmatrix}=-Z^{\mathrm{T}}(x)g(x), \qquad A^{\mathrm{T}}(x)\begin{bmatrix} z^2-y \\ y-z \end{bmatrix}=-c(x) \tag{2.7}$$

for all $x=(y, z)^{\mathrm{T}}$ such that $y=z$ or $y=z^2$ in a small neighbourhood of x^*. If $c(y, z)=y+\bar{c}(y, z)$, then the second equation of (2.7) is satisfied for $y=z$ and $y=z^2$ if

$$-\frac{\partial \bar{c}(z, z)}{\partial y}(z^2-z)=\bar{c}(z, z)+z^2, \qquad -\frac{\partial \bar{c}(z^2, z)}{\partial z}(z^2-z)=\bar{c}(z^2, z)+z^2. \tag{2.8}$$

Let $\bar{c}(y, z)$ have the form

$$\bar{c}(y, z)=D\left(\frac{z-y}{1-z}, \frac{y-z^2}{1-z}\right), \tag{2.9}$$

in order that (2.8) reduces to

$$D(0, z)+z^2 = z\left[-\frac{\partial}{\partial y} D(0, z)+\frac{\partial}{\partial z} D(0, z)\right],$$
$$D(z, 0)+z^2 = z\left[(1+z)\frac{\partial}{\partial y} D(z, 0)-2z\frac{\partial}{\partial z} D(z, 0)\right], \tag{2.10}$$

where $\partial/\partial y$ and $\partial/\partial z$ are the partial derivatives with respect to the first and second variable respectively. One can easily verify that

$$D(\eta, \zeta)=\eta^2+\eta\zeta+2\zeta^2 \tag{2.11}$$

solves (2.10). Therefore, if we let

$$c(y, z)=y+\frac{1}{(1-z^2)}[(z-y)^2+(z-y)(y-z^2)+2(y-z^2)^2], \tag{2.12}$$

we satisfy the required conditions that depend on the second part of eq. (2.7).

To complete the construction of the example, we need to show the existence of $f(x)$ such that the first equation of (2.7) holds for $y=z$ and $y=z^2$, so we require

$$\frac{\partial}{\partial z} c(z, z)\frac{\partial}{\partial y} f(z, z)-\frac{\partial}{\partial y} c(z, z)\frac{\partial}{\partial z} f(z, z) = -\frac{\partial}{\partial z} c(z, z)(z^2-z),$$
$$\frac{\partial}{\partial z} c(z^2, z)\frac{\partial}{\partial y} f(z^2, z)-\frac{\partial}{\partial y} c(z^2, z)\frac{\partial}{\partial z} f(z^2, z) = \frac{\partial}{\partial y} c(z^2, z)(z^2-z). \tag{2.13}$$

Since (2.12) gives the values

$$\frac{\partial}{\partial y} c(z, z)=\frac{(1+2z)}{(1-z)},\qquad \frac{\partial}{\partial z} c(z, z)=z\frac{(1-4z)}{(1-z)}$$
$$\frac{\partial}{\partial y} c(z^2, z)=\frac{(1-2z)}{(1-z)},\qquad \frac{\partial}{\partial z} c(z^2, z)=\frac{2z}{(1-z)}, \tag{2.14}$$

we let $f(y, z)=\frac{1}{2}z^2-yz+\bar{f}(y, z)$, in order that (2.13) is equivalent to

$$-z(1-4z)\frac{\partial}{\partial y}\bar{f}(z, z)+(1+2z)\frac{\partial}{\partial z}\bar{f}(z, z)=-z^2(1-4z)(2-z),$$
$$-2z\frac{\partial}{\partial y}\bar{f}(z^2, z)+(1-2z)\frac{\partial}{\partial z}\bar{f}(z^2, z)=-2z^2. \tag{2.15}$$

Corresponding to (2.9), we let $\bar{f}(y, z)$ have the form

$$\bar{f}(y, z) = G\left(\frac{z-y}{1-z}, \frac{y-z^2}{1-z}\right). \tag{2.16}$$

It follows that (2.15) reduces to

$$\begin{aligned} &(1+4z)\frac{\partial}{\partial y} G(0, z) - 2z\frac{\partial}{\partial z} G(0, z) = -z^2(1-4z)(2-z), \\ &(1+2z)\frac{\partial}{\partial y} G(z, 0) - 4z\frac{\partial}{\partial z} G(z, 0) = -2z^2. \end{aligned} \tag{2.17}$$

Because the function

$$G(\eta, \zeta) = -\tfrac{2}{3}\eta^3 - \eta^2\zeta - 2\eta\zeta^2 - \tfrac{17}{6}\zeta^3 + \tfrac{1}{2}\zeta^4 \tag{2.18}$$

satisfies these equations, we let

$$\begin{aligned} f(x) = \tfrac{1}{2}z^2 - yz + \frac{1}{6(1-z)^3}\bigg[&-4(z-y)^3 - 6(z-y)^2(y-z^2) \\ &- 12(z-y)(y-z^2)^2 - 17(y-z^2)^3 + 3\frac{(y-z^2)^4}{1-z}\bigg]. \end{aligned} \tag{2.19}$$

Thus we also obtain the required conditions that depend on the first part of expression (2.7).

For $f(x)$ and $c(x)$ defined by (2.19) and (2.12), it is easy to show that (2.1) and (2.3) hold; further $Y(x)$, $Z(x)$ and $R(x)$ defined by (2.6) are continuously differentiable in a small neighbourhood of x^*. Since $[A(x), Z(x)]$ is nonsingular, it follows from (2.7) that (2.4) holds for all k provided that $\|x_k\|_2$ is very small and that x_k satisfies either $y_k = z_k$ or $y_k = z_k^2$. This completes the construction of the example.

The example shows that, although the reduced Hessian method has 2-step Q-superlinear convergence, the rate of convergence on alternate iterations may be only linear. Further, it is not even linear in the sense that $\|x_{2k+1}\|_\infty = \|x_{2k}\|_\infty$ for sufficiently large k.

3. Remarks

A computer program has been written to test whether the phenomenon exposed by the given example occurs in practice or not. Since computing errors may cause the iterates not to remain on the two curves $y = z$ and $y = z^2$, it is reasonable for one to think that the theoretical analysis of the above example may not hold in practice. However, we find that the one-fast-one-slow convergence phenomenon occurs not only for the initial points on the two curves $y = z$ and $y = z^2$, but also for many other initial points. For example the five initial points $(0.1, 0.1)^T$, $(0.1, 0.2)^T$, $(0.2, 0.1)^T$, $(0.0, 0.1)^T$, and $(0.1, 0.0)^T$, and $B_k = 1$ for all k give the values of $r_k = \|x_k\|_\infty / \|x_{k-1}\|_\infty$ that are shown in Table 1. Due to rounding errors the method terminates for all five initial points, the test for convergence being $\|Z^T(x)g(x)\|_\infty + \|c(x)\|_\infty = 0$. These results indicate that only the second initial point $(0.1, 0.2)^T$ gives Q-superlinear convergence.

Table 1

k	$\{r_k\}$ evaluated with different initial points:				
	0.1 0.1	0.1 0.2	0.2 0.1	0.0 0.1	0.1 0.0
1	1	0.304	1.35	0.141	1.28
2	0.1(0)	0.199	0.394	1.02	0.139
3	1	0.160	0.662	0.262(−1)	0.835
4	0.1(−1)	0.453(−1)	0.728(−1)	0.532	0.148(−1)
5	1	0.374(−1)	0.959	0.202(−3)	1
6	0.1(−3)	0.210(−2)	0.493(−2)	1	0.250(−3)
7	1	0.156(−2)	1	0.457(−7)	0.877
8	0.1(−7)	0.443(−5)	0.253(−4)	0.894	0.482(−7)
9	1	0.244(−5)	0.962	0.167(−14)	1
10	0.1(−15)	0.196(−10)	0.594(−9)	1	0.254(−14)
11	1	0.598(−11)	1	0.280(−29)	0.914
12	0.1(−31)	0.0	0.353(−18)	1	0.541(−29)
13	1		1	0.0	1
14	0.0		0.125(−38)		0.0
15			1		
16			0.0		

We also tried 25 initial points which have the form $((\eta - 0.5)/10, (\zeta - 0.5)/10)^T$, where η, ζ are random numbers in [0, 1]. It turns out that the Q-superlinear convergence occurs in 14 cases. For the other 11 cases, the one-fast-one-slow behaviour occurs, and it is usual for alternate iterates to get closer to the two curves $y = z$ and $y = z^2$ as k increases.

The method is a local one that may fail to converge unless $\|x_0\|$ is sufficiently small. For example the initial point $(0.01, 0.5)^T$ was found to be too far from the solution.

Though $\lambda^* = 0$ in Section 2, it is easy to show that examples with $\lambda^* \neq 0$ can be constructed, due to the following lemma:

Lemma 3.1. *Let u_i $(i = 1, 2, \ldots, m)$ be any m constants. The method given in Section 1 generates the same sequence $\{x_k\}$ if the objective function is altered from $f(x)$ to $f(x) + \sum_{i=1}^m u_i c_i(x)$, provided that the initial point and $\{B_k; k = 1, 2, 3, \ldots\}$ are the same in both cases.*

Proof. Let $\{x_k\}$ be the sequence generated by the method when the objective function is $f(x)$ and $\{\bar{x}_k\}$ be that when the objective function is $f(x) + u^T c(x)$, where $u = (u_1, \ldots, u_m)^T$.

If $x_k = \bar{x}_k$, then the method defines x_{k+1} and $\bar{x}_{k+1}$ by the equations

$$\begin{bmatrix} B_k Z^{\mathrm{T}}(x_k) \\ A^T(x_k) \end{bmatrix} (x_{k+1} - x_k) = -\begin{bmatrix} Z^{\mathrm{T}}(x_k) g(x_k) \\ c(x_k) \end{bmatrix} \tag{3.1}$$

and

$$\begin{bmatrix} B_k Z^{\mathrm{T}}(x_k) \\ A^T(x_k) \end{bmatrix} (\bar{x}_{k+1} - \bar{x}_k) = -\begin{bmatrix} Z^{\mathrm{T}}(x_k)[g(x_k) + A(x_k) u] \\ c(x_k) \end{bmatrix}. \tag{3.2}$$

Since $Z^{\mathrm{T}}(x_k) A(x_k) = 0$, it follows from (3.1) and (3.2) that

$$\bar{x}_{k+1} - \bar{x}_k = x_{k+1} - x_k. \tag{3.3}$$

Therefore $\bar{x}_{k+1} = x_{k+1}$. By induction, since $x_0 = \bar{x}_0$, the lemma is true.

In the example given in Section 2, we have that $\|x_{2k+1} - x^*\|_\infty = \|x_{2k} - x^*\|_\infty$ for all k. We believe that by modifying the example it is possible to show that alternate terms of the sequence $\{\|x_k - x^*\|, k = 1, 2, 3, \ldots\}$ may increase for any choice of norm.

Since our example shows that algorithm (1.5)-(1.6) may converge only 2-step Q-superlinearly even if $B_k \equiv Z^{\mathrm{T}}(x^*) W^* Z(x^*)$, it is not possible to obtain an 1-step Q-superlinear convergence resut for (1.5)-(1.6) if B_k is some kinds of approximation of $Z^{\mathrm{T}}(x^*) W^* Z(x^*)$. Pointed out by the referee, a simlar example is given by Byrd [1] where B_k is the reduced Hessian at the current iterate instead of at the solution.

The following analysis is motivated by Goodman [2], Nocedal and Overton [3], Overton [4] and Stoer [6]. Suppose $B_k = B(x_k)$ where $B(x)$ from $\mathbb{R}^m$ to $\mathbb{R}^m$ is continuously differentiable in a neighbourhood of the solution x^* and $B(x^*)$ is nonsingular. (1.5)-(1.6) is Newton's method for the fixed-point problem:

$$x = \phi(x)$$

where

$$\begin{aligned}
\phi(x) &= x - \begin{bmatrix} B(x) Z^{\mathrm{T}}(x) \\ A^{\mathrm{T}}(x) \end{bmatrix}^{-1} \begin{bmatrix} Z^{\mathrm{T}}(x) g(x) \\ c(x) \end{bmatrix} \\
&= x - [Z(x) B^{-1}(x) A(x) (A^{\mathrm{T}}(x) A(x))^{-1}] \begin{bmatrix} Z^{\mathrm{T}}(x) g(x) \\ c(x) \end{bmatrix} \\
&= x - Z(x) B^{-1}(x) Z^{\mathrm{T}}(x)[g(x) - A(x)\lambda^*] - A(x)[A^{\mathrm{T}}(x) A(x)]^{-1} c(x).
\end{aligned}$$

Because $g(x^*) - A(x^*)\lambda^* = 0$ and $c(x^*) = 0$, it follows that

$$D\phi(x^*) = I - Z(x^*) B^{-1}(x^*) Z^{\mathrm{T}}(x^*) W^* - A(x^*)[A^{\mathrm{T}}(x^*) A(x^*)]^{-1} A^{\mathrm{T}}(x^*).$$

Using $Q^{\mathrm{T}}(x^*) Q(x^*) = I$, we have that

$$D\phi(x^*) = Q(x^*)$$

$$\times \begin{bmatrix} 0 & 0 \\ -B^{-1}(x^*) Z^{\mathrm{T}}(x^*) W^* Y(x^*) & I - B^{-1}(x^*) Z^{\mathrm{T}}(x^*) W^* Z(x^*) \end{bmatrix} Q^{\mathrm{T}}(x^*).$$

Let

$$\rho_1 = \rho(B^{-1}(x^*)Z^{\mathrm{T}}(x^*)W^*Y(x^*)),$$

$$\rho_2 = \rho(I - B^{-1}(x^*)Z^{\mathrm{T}}(x^*)W^*Z(x^*)),$$

where $\rho(\cdot)$ is the spectral redius of a matrix. It is evident that

$$\rho(D\phi(x^*)) \leqslant \rho_1 + \rho_2, \qquad \rho([D\phi(x^*)]^2) \leqslant \rho_2(\rho_1 + \rho_2).$$

Hence if $\rho_2(\rho_1+\rho_2)<1$, x_k converges to x^* locally and 2-step linearly. Specially, when $B(x^*) = Z^{\mathrm{T}}(x^*)W^*Z(x^*)$, we have that $\rho_2 = 0$. In this case $\rho([D\phi(x^*)]^2) = 0$, consequently the algorithm converges locally and 2-step superlinearly. If $Z^{\mathrm{T}}(x^*)W^*Y(x^*) \neq 0$, we have that

$$\rho(D\phi(x^*)) \geqslant \rho_1 \geqslant \frac{\rho(Z^{\mathrm{T}}(x^*)W^*Y(x^*))}{\rho(B(x^*))} > 0.$$

That is why the algorithm may converge only 2-step superlinearly instead of 1-step superlinarly. It is also noted that the algorithm converges 1-step superlinearly if $B(x^*) = Z^{\mathrm{T}}(x^*)W^*Z(x^*)$ and $Z^{\mathrm{T}}(x^*)W^*Y(x^*) = 0$.

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