An interior-point trust-region polynomial algorithm for convex quadratic minimization subject to general convex constraints

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Abstract. An interior-point trust-region algorithm is proposed for minimization of a convex quadratic objective function over a general convex set. The algorithm uses a trust-region model to ensure descent on a suitable merit function. The complexity of our algorithm is proved to be as good as the interior-point polynomial algorithm.

Key words. interior-point algorithm, self-concordant barrier, trust-region subproblem.

1 Introduction

The idea of interior-point trust-region algorithm can be traced back to Dikin[3] where an interior ellipsoid method was developed for linear problems. Recently, Tseng[8] produced a global and local convergence analysis of Dikin's algorithm for indefinite quadratic programming. We also refer Absil and Tits[1] for this direction. Ye[9] developed an affine scaling algorithm for indefinite quadratic programming by solving sequential trust-region subproblem. Global first-order and second-order convergence results were proved, and later enhanced by Sun[7] for the convex case. In the trust-region literature, Conn, Gould and Toint [2] developed a primal barrier trust-region algorithm, which has been recently extended to solve symmetric cone

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programming by Lu and Yuan[4]. In this paper, we present an affine-scaling primal barrier interior-point trust-region algorithm for minimizing a convex quadratic objective function over a general convex set. To our knowledge, our algorithm is the first interior-point trust-region algorithm for this special convex programming, by using the techniques and properties in both interior-point algorithms and trust-region methods. We show that the complexity of our algorithm is as good as the standard interior-point polynomial algorithms. This provides strong theoretical supports to the good practical performance of the interior-point trust-region algorithm given by Lu and Yuan[4]. Although our algorithm is based on a fixed trust-region radius and solves the trust-region subproblem exactly, the framework of the interior-point trust-region algorithm allows us to make the trust-region radius flexible and use iterative methods to solve the trust-region subproblem approximately in practical implementation. This advantage makes the interior-point trust-region algorithm competitive with the pure interior-point algorithm for solving large-scale problems. The goal of this paper is to show that the complexity of interior-point trust-region algorithm is as good as the complexity of pure (standard) interior-point algorithm in convex programming.

2 Self-concordant barrier and its properties

In this section, we present the concept of self-concordant barrier and its properties that will play an important in our analysis of section 3.

The following definition is due to Nesterov and Nemirovskii[5].

Definition 2.1. *Let $F : K^\circ \to R$ be a C^3-smooth convex function such that $F(x) \to \infty$ as $x \in K^\circ$ approaches the boundary of K and*

$$| F'''(x)[h, h, h] | \leq 2 \ \langle F''(x)h, h\rangle^{3/2} \tag{2.1}$$

for all $x \in K^\circ$ and for all $h \in E$. Then F is called a self-concordant function for K. F is called a self-concordant barrier if F is a self concordant function and

$$\vartheta := sup_{x\in K^\circ}\langle F'(x), F''(x)^{-1}F'(x)\rangle < \infty. \tag{2.2}$$

ϑ is called barrier parameter of F.

Let $F''(x)$ denote the Hessian of a self-concordant function $F(x)$. Since it is positive definite, for every $x \in K^\circ$, $\|v\|_x = \langle v, F''(x)v\rangle^{\frac{1}{2}}$ is a norm on

E induced by $F''(x)$. Let $B_x(y, r)$ denote the open ball of radius r centered at y, where the radius is measured w.r.t. $\| \|_x$. This ball is called the Dikin ball. The following lemmas are very crucial for the analysis of our algorithm in the next section. For the proofs, see e.g. the chapter 2 of Renegar[6].

Lemma 2.1. *Assume $F(x)$ is a self-concordant function for K, then for all $x \in K^\circ$, we have $B_x(x, 1) \subseteq K^\circ$ and if whenever $y \in B_x(x, 1)$ we have*

$$\frac{\|v\|_y}{\|x\|_x} \leq \frac{1}{1 - \|y - x\|_x} \ for\ all\ v \neq 0. \tag{2.3}$$

Lemma 2.2. *Assume $F(x)$ is a self-concordant function for K, $x \in K^\circ$ and $y \in B_x(x, 1)$, then*

$$|F(y) - F(x) - \langle F'(x), y - x\rangle - \frac{\langle y - x, F''(x)(y - x)\rangle}{2}| \leq \frac{\|y - x\|_x^3}{3(1 - \|y - x\|_x)}. \tag{2.4}$$

Let $n(x) := -F''(x)^{-1}F'(x)$ be the Newton step of $F(x)$.

Lemma 2.3. *Assume $F(x)$ is a self-concordant function. If $\|n(x)\|_x \leq \frac{1}{4}$ then $F(x)$ has a minimizer z and*

$$\|z - x\|_x \leq \|n(x)\|_x + \frac{3\|n(x)\|_x^2}{(1 - \|n(x)\|_x)^3}. \tag{2.5}$$

Lemma 2.4. *Assume $F(x)$ is a self-concordant barrier with barrier parameter ϑ. If $x, y \in K^\circ$ then*

$$\langle F'(x), y - x\rangle \leq \vartheta. \tag{2.6}$$

3 The interior-point trust-region algorithm

In this section, we present our algorithm and give the complexity analysis.

We consider the following optimization problem

$$\min \quad q(x) = \frac{1}{2}\langle x, Qx\rangle + \langle c, x\rangle \tag{3.1}$$

$$\text{subject to} \quad x \in K. \tag{3.2}$$

Here $Q : E \mapsto E$ is a positive definite or positive semi-definite linear operator, $c \in E$. K is a bounded convex set with nonempty relative interior.

We assume $F(x)$ is the self-concordant barrier for K and define the merit function as

$$f_{\eta_k}(x) = \eta_k q(x) + F(x). \tag{3.3}$$

From definition 2.1, we can see that function $f_{\eta_k}(x)$ itself is also a self-concordant function. We want to decrease the value of $f_{\eta_k}(x)$ for a fixed η_k in each inner iteration, and increase η_k to positive infinity in outer iterations. From Lemma 2.1, for any $x_{k,j} \in K^\circ$ and $d \in E$, we have that $x_{k,j} + d \in K^\circ$ provided that $\|F''(x_{k,j})^{\frac{1}{2}} d\| \leq \alpha_{k,j} < 1$. It follows from Lemma 2.2 that

$$\begin{aligned} F(x_{k,j}+d) - F(x_{k,j}) &\leq \langle F'(x_{k,j}), d\rangle + \frac{\langle d, F''(x_{k,j})d\rangle}{2} + \frac{\|d\|^3_{x_{k,j}}}{3(1-\|d\|_{x_{k,j}})} \\ &\leq \langle F'(x_{k,j}), d\rangle + \frac{\langle d, F''(x_{k,j})d\rangle}{2} + \frac{\alpha^3_{k,j}}{3(1-\alpha_{k,j})} \end{aligned} \tag{3.4}$$

Therefore, we get

$$\begin{aligned} f_{\eta_k}(x_{k,j}+d) \quad - \quad f_{\eta_k}(x_{k,j}) &\leq \frac{\langle d, (\eta_k Q + F''(x_{k,j}))d\rangle}{2} \\ &+ \langle \eta_k(Qx_{k,j}+c) + F'(x_{k,j}), d\rangle + \frac{\alpha^3_{k,j}}{3(1-\alpha_{k,j})}. \end{aligned} \tag{3.5}$$

The above inequality gives an up bound for $f_{\eta_k}(x_{k,j}+d)$. It is natural to minimize this bound in order to force a decrease on the function $f_{\eta_k}(.)$. This leads to the following trust-region subproblem

$$\min \quad \tfrac{1}{2}\langle d, (\eta_k Q + F''(x_{k,j}))d\rangle + \langle \eta_k(Qx_{k,j}+c) + F'(x_{k,j}), d\rangle = m_{k,j}(d) \tag{3.6}$$

$$s.t. \quad \|F''(x_{k,j})^{\frac{1}{2}} d\|^2 \leq \alpha^2_{k,j}. \tag{3.7}$$

Define

$$Q_{k,j} = \eta_k F''(x_{k,j})^{-\frac{1}{2}} Q F''(x_{k,j})^{-\frac{1}{2}} + I, \tag{3.8}$$

$$c_{k,j} = F''(x_{k,j})^{-\frac{1}{2}} \left(\eta_k(Qx_{k,j}+c) + F'(x_{k,j})\right), \tag{3.9}$$

and using the transformation

$$d' = F''(x_{k,j})^{\frac{1}{2}} d, \tag{3.10}$$

we can rewrite subproblem (3.6)-(3.7) into

$$\min \quad q'_{k,j}(d') = \tfrac{1}{2}\langle d', Q_{k,j} d'\rangle + \langle c_{k,j}, d'\rangle \tag{3.11}$$

$$\|d'\|^2 \leq \alpha_{k,j}^2. \tag{3.12}$$

Once $d'_{k,j}$ is computed, we obtain the step

$$d_{k,j} = F''(x_{k,j})^{-\frac{1}{2}} d'_{k,j}, \tag{3.13}$$

and it follows from inequality (3.5) that

$$f_{\eta_k}(x_{k,j} + d_{k,j}) \quad - \quad f_{\eta_k}(x_{k,j}) \leq q'_{k,j}(d'_{k,j}) + \frac{\alpha_{k,j}^3}{3(1-\alpha_{k,j})}. \tag{3.14}$$

Algorithm 3.1. *(An Interior-Point Trust Region Algorithm)*

Step 0 **Initialization.** *An initial point $x_{0,0} \in K^\circ$ and an initial parameter $\eta_0 > 0$ are given. Set $\alpha_{k,j} = \alpha < 1$ for some constant α. Set $k = 0$ and $j = 0$.*

Step 1 **Test inner iteration termination.** *If*

$$\langle c_{k,j}, Q_{k,j}^{-1} c_{k,j}\rangle \leq \frac{1}{9}, \tag{3.15}$$

set $x_{k+1,0} = x_{k,j}$ and go to Step 3.

Step 2 **Step calculation.** *Solve problem (3.11)-(3.12) obtaining $d'_{k,j}$ exactly, set $d_{k,j}$ by (3.13) and $x_{k,j+1} = x_{k,j} + d_{k,j}$.*

Step 3 **Update parameter η.** *Set $\eta_{k+1} = \theta\eta_k$ for some constant $\theta > 1$. Increase k by 1 and go to step 1.*

Theorem 3.1. *a) If we choose $\alpha = \frac{1}{4}$, we have that*

$$f_{\eta_k}(x_{k,j+1}) - f_{\eta_k}(x_{k,j}) < -\frac{1}{48}, \tag{3.16}$$

which is independent of k and j.
b)If the initial point $x_{0,0}$ satisfies the condition (3.15), for any $\epsilon > 0$, our algorithm obtains a solution x which satisfies $q(x) - q(x^*) < \epsilon$ in at most

$$\frac{48\theta(\vartheta + \sqrt{\vartheta}) \ln \frac{\vartheta+\sqrt{\vartheta}}{\epsilon\eta_0}}{\ln\theta} \tag{3.17}$$

steps, here $x^* = argmin_{x\in K} q(x)$.

To prove part *a*) of this theorem, we need the following lemma which is well-known in trust-region literature.

Lemma 3.1. *Any global minimizer $d'_{k,j}$ of problem (3.11)-(3.12) satisfies the equation*

$$(Q_{k,j} + \mu_{k,j} I) d'_{k,j} = -c_{k,j}, \tag{3.18}$$

here $Q_{k,j} + \mu_{k,j} I$ is positive semi-definite, $\mu_{k,j} \geq 0$ and $\mu_{k,j}(\|d'_{k,j}\| - \alpha_{k,j}) = 0$.

For a proof, see e.g. Section 7.2 of Conn, Gould and Toint[2].

Proof of Theorem 3.1 part a). If the solution of (3.11)-(3.12) lies on the boundary of the trust-region, that is, $\|d'_{k,j}\| = \alpha_{k,j}$, we have

$$\begin{aligned}
q'_{k,j}(d'_{k,j}) &= \frac{1}{2}\langle d'_{k,j}, Q_{k,j} d'_{k,j}\rangle + \langle c_{k,j}, d'_{k,j}\rangle \\
&= \langle d'_{k,j}, Q_{k,j} d'_{k,j} + c_{k,j}\rangle - \frac{1}{2}\langle d'_{k,j}, Q_{k,j} d'_{k,j}\rangle \\
&= \langle d'_{k,j}, -\mu_{k,j} d'_{k,j}\rangle - \frac{1}{2}\langle d'_{k,j}, (\eta_k F''(x_{k,j})^{-\frac{1}{2}} Q F''(x_{k,j})^{-\frac{1}{2}} + I) d'_{k,j}\rangle \\
&= -\mu_{k,j}\alpha_{k,j}^2 - \frac{1}{2}\langle d'_{k,j}, \eta_k F''(x_{k,j})^{-\frac{1}{2}} Q F''(x_{k,j})^{-\frac{1}{2}} d'_{k,j}\rangle - \frac{1}{2}\alpha_{k,j}^2 \\
&\leq -\frac{1}{2}\alpha_{k,j}^2 = -\frac{1}{32}.
\end{aligned}$$

In the above, the third equality follows from the equalities (3.8) and (3.18), and the inequality follows from the fact that $\eta_k F''(x_{k,j})^{-\frac{1}{2}} Q F''(x_{k,j})^{-\frac{1}{2}}$ is positive definite or positive semi-definite. Therefore, it follows from inequality (3.14) that

$$f_{\eta_k}(x_{k,j+1}) - f_{\eta_k}(x_{k,j}) \leq -\frac{1}{32} + \frac{(\frac{1}{4})^3}{3(1-\frac{1}{4})} < -\frac{1}{48}.$$

If the solution of (3.11)-(3.12) lies in the interior of the trust-region, that is, $\|d'_{k,j}\| < \alpha_{k,j}$, from Lemma 3.1 we know $\mu_{k,j} = 0$ and consequently $d'_{k,j} = -Q_{k,j}^{-1} c_{k,j}$ which gives that

$$q'_{k,j}(d'_{k,j}) = \frac{1}{2}\langle d'_{k,j}, Q_{k,j} d'_{k,j}\rangle + \langle c_{k,j}, d'_{k,j}\rangle = -\frac{1}{2}\langle c_{k,j}, Q_{k,j}^{-1} c_{k,j}\rangle. \tag{3.19}$$

By the mechanism of our algorithm, we know that $\langle c_{k,j}, Q_{k,j}^{-1} c_{k,j}\rangle > \frac{1}{9}$ for all k and j. Therefore,

$$f_{\eta_k}(x_{k,j+1}) - f_{\eta_k}(x_{k,j}) \leq -\frac{1}{18} + \frac{(\frac{1}{4})^3}{3(1-\frac{1}{4})} < -\frac{1}{48}.$$

Let $n_{\eta_k}(x_{k,j})$ be the Newton step of $f_{\eta_k}(x)$ at the point $x_{k,j}$. We should point out that

$$\begin{aligned}
\|n_{\eta_k}(x_{k,j})\|_{x_{k,j}} &= \langle f'_{\eta_k}(x_{k,j}), f''_{\eta_k}(x_{k,j})^{-1} f'_{\eta_k}(x_{k,j})\rangle \\
&= \langle \eta_k(Qx_{k,j}+c)+F'(x_{k,j}), (\eta_k Q + F''(x_{k,j}))^{-1}(\eta_k(Qx_{k,j}+c)+F'(x_{k,j}))\rangle \\
&= \langle c_{k,j}, Q_{k,j}^{-1} c_{k,j}\rangle,
\end{aligned}$$

where the last equality follows equalities (3.8) and (3.9). This equality connects equality (3.19) and the assumption of the following two lemmas, which tells us that we can stop the inner iteration if the reduction of the objective function with an interior solution is smaller than some constant. The following two lemmas extend the results of Renegar[6] for minimizing a linear objective function over a convex set.

Lemma 3.2. *Let* $x^* = argmin_{x\in K} q(x)$. *If* $\|n_\eta(x)\|_x \le \frac{1}{9}$, *then*

$$q(x) - q(x^*) \le \frac{\vartheta + \sqrt{\vartheta}}{\eta}. \tag{3.20}$$

Proof. Let $x(\eta) = argmin_{x\in K} f_\eta(x)$. Then

$$\begin{aligned}
q(x(\eta)) - q(x^*) &\le \langle q'(x(\eta)), x(\eta) - x^*\rangle \\
&= \langle \frac{-F'(x(\eta))}{\eta}, x(\eta) - x^*\rangle \le \frac{\vartheta}{\eta}.
\end{aligned} \tag{3.21}$$

The first inequality follows from the convexity of $q(x)$. The equality follows from the fact that $f'(x(\eta)) = 0$, and the last inequality follows from Lemma 2.4.

It easily follows from Lemma 2.3 that

$$\|x - x(\eta)\|_x \le \frac{1}{9} + \frac{3(\frac{1}{9})^2}{(1-\frac{1}{9})^3} < \frac{1}{4} \tag{3.22}$$

and consequently from Lemma 2.1 that

$$\|x - x(\eta)\|_{x(\eta)} \le \frac{\|x - x(\eta)\|_x}{1 - \|x - x(\eta)\|_x} < \frac{1}{3}. \tag{3.23}$$

Then we have

$$
\begin{aligned}
q(x) - q(x(\eta)) &= \langle q'(x(\eta)), x - x(\eta)\rangle + \frac{\langle x - x(\eta), Q(x - x(\eta))\rangle}{2} \\
&= \langle \frac{-F'(x(\eta))}{\eta}, x - x(\eta)\rangle + \frac{\langle x - x(\eta), \eta Q(x - x(\eta))\rangle}{2\eta} \\
&\leq \frac{\langle -F''(x(\eta))^{-\frac{1}{2}} F'(x(\eta)), F''(x(\eta))^{\frac{1}{2}}(x - x(\eta))\rangle}{\eta} + \frac{\|x - x(\eta)\|_x}{2\eta} \\
&\leq \frac{\|F''(x(\eta))^{-\frac{1}{2}} F'(x(\eta))\| \|F''(x(\eta))^{\frac{1}{2}}(x - x(\eta))\|}{\eta} + \frac{1}{8\eta} \\
&\leq \frac{\sqrt{\vartheta}\|x - x(\eta)\|_{x(\eta)}}{\eta} + \frac{1}{8\eta} \leq \frac{\sqrt{\vartheta}}{3\eta} + \frac{1}{8\eta} \leq \frac{\sqrt{\vartheta}}{\eta}, \qquad (3.24)
\end{aligned}
$$

where the last inequality follows from the fact that ϑ is always greater than 1 and the third last inequality uses the definition of ϑ. By adding the inequality (3.21) and inequality (3.24), we get inequality (3.20). □

The above lemma tells us that to get an ϵ-solution, we only need

$$
\eta_k = \eta_0 \theta^k \geq \frac{\vartheta + \sqrt{\vartheta}}{\epsilon},
$$

which is true provided that the number of outer iterations k satisfies

$$
k \geq \frac{\ln \frac{\vartheta + \sqrt{\vartheta}}{\epsilon \eta_0}}{\ln \theta}. \qquad (3.25)
$$

Lemma 3.3. *If* $\|n_{\eta_k}(x)\|_x \leq \frac{1}{9}$, *then*

$$
f_{\eta_{k+1}}(x) - f_{\eta_{k+1}}(x(\eta_{k+1})) \leq \theta(\vartheta + \sqrt{\vartheta}). \qquad (3.26)
$$

Proof. From the convexity of $f_{\eta_{k+1}}(x)$ and inequality (3.22), we can show that

$$
\begin{aligned}
f_{\eta_{k+1}}(x) &- f_{\eta_{k+1}}(x(\eta_k)) \le \langle f'_{\eta_{k+1}}(x), x - x(\eta_k)\rangle \\
&= \langle \eta_{k+1}(Qx+c) + F'(x), x - x(\eta_k)\rangle \\
&= \frac{\eta_{k+1}}{\eta_k}\langle \eta_k(Qx+c) + F'(x), x - x(\eta_k)\rangle + (\frac{\eta_{k+1}}{\eta_k} - 1)\langle F'(x), x(\eta_k) - x\rangle \\
&= \theta\langle f''^{-\frac{1}{2}}_{\eta_k}(x)(\eta_k(Qx+c) + F'(x)), f''_{\eta_k}(x)^{\frac{1}{2}}(x - x(\eta_k))\rangle \\
&\quad +(\theta-1)\langle F''(x)^{-\frac{1}{2}}F'(x), F''(x)^{-\frac{1}{2}}(x(\eta_k) - x)\rangle \\
&\le \theta\|f''^{-\frac{1}{2}}_{\eta_k}(x)(\eta_k(Qx+c) + F'(x))\|\|f''_{\eta_k}(x)^{\frac{1}{2}}(x - x(\eta_k))\| \\
&\quad +(\theta-1)\|F''(x)^{-\frac{1}{2}}F'(x)\|\|F''(x)^{\frac{1}{2}}(x(\eta_k) - x)\| \\
&\le \theta\|n_{\eta_k}(x)\|_x\|x(\eta_k) - x\|_x + (\theta-1)\sqrt{\vartheta}\|x(\eta_k) - x\|_x \\
&\le \theta\frac{1}{9}\frac{1}{4} + (\theta-1)\sqrt{\vartheta}\frac{1}{4} \le \theta\sqrt{\vartheta}. \qquad (3.27)
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
f_{\eta_{k+1}}(x(\eta_k)) &- f_{\eta_{k+1}}(x(\eta_{k+1})) \le \langle f'_{\eta_{k+1}}(x(\eta_k)), x(\eta_k) - x(\eta_{k+1})\rangle \\
&= \langle \eta_{k+1}(Qx(\eta_k)+c) + F'(x(\eta_k)), x(\eta_k) - x(\eta_{k+1})\rangle \\
&= \frac{\eta_{k+1}}{\eta_k}\langle \eta_k(Qx(\eta_k)+c) + F'(x(\eta_k)), x(\eta_k) - x(\eta_{k+1})\rangle \\
&\quad +(\frac{\eta_{k+1}}{\eta_k} - 1)\langle F'(x(\eta_k)), x(\eta_{k+1}) - x(\eta_k)\rangle \\
&= \theta\langle f'_{\eta_k}(x(\eta_k)), x(\eta_k) - x(\eta_{k+1})\rangle \\
&\quad +(\theta-1)\langle F'(x(\eta_k)), x(\eta_{k+1}) - x(\eta_k)\rangle \\
&= (\theta-1)\langle F'(x(\eta_k)), x(\eta_{k+1}) - x(\eta_k) < \theta\vartheta, \qquad (3.28)
\end{aligned}
$$

where the last inequality follows from Lemma 2.4 and the last equality follows from the fact that $x(\eta_k)$ minimizes $f_{\eta_k}(x)$ (which implies that $f'_{\eta_k}(x(\eta_k)) = 0$). By adding inequality (3.27) and inequality (3.28), we get inequality (3.26). □

This lemma and Part a) of Theorem 3.1 tell us that we need at most

$$48(\vartheta + \sqrt{\vartheta}) \qquad (3.29)$$

steps in each inner iteration.

Proof of Theorem 3.1 part b). It follows from (3.25) and (3.29). □

Finally, a brief discussion is given on the case when the initial point $x_{0,0}$ does not satisfy condition (3.15). Without loss of generality, we can start from the analytic center of the feasible set K since K is a bounded convex set. Let $x(\eta_0) = argmin_{x\in K} f_{\eta_0}(x)$ and $x^* = argmin_{x\in K} q(x)$. Since $x_{0,0} = argmin_{x\in K} F(x)$, we have

$$f_{\eta_0}(x_{0,0}) - f_{\eta_0}(x(\eta_0)) \leq \eta_0(q(x_{0,0}) - q(x^*)). \tag{3.30}$$

By choosing $\eta_0 \leq \frac{1}{q(x_{0,0})-q(x^*)}$, we can see that Theorem 3.1 part a) implies that condition (3.15) will be satisfied after at most 48 steps.

References

[1] P.A. Absil and A.L. Tits., 2007, Newton-KKT Interior-Point Methods for Indefinite Quadratic Programming, Computational Optimization and Applications. 36, 5-41.

[2] A.R. Conn, N.I.M. Gould and Ph.L. Toint., 2000, Trust-region Methods, SIAM Publications, Philadelphia, Pennsylvania.

[3] I. I. Dikin., 1967, Iterative solution of problems of linear and quadratic programming. Soviet Math. Doklady. 8, 674-675.

[4] Y. Lu and Y. Yuan., 2007, An interior-point trust-region algorithm for general symmtric cone programming, SIAM Journal on Optimization. 18, 65-86.

[5] Y.E. Nesterov and A.S. Nemirovskii.,1994, Interior-point Polynomial Algorithms in Convex Programming, SIAM Publications, Philadelphia, Pennsylvania.

[6] J. Renegar., 2001, A Mathematical View of Interior-point Methods in Convex Programming, SIAM Publications, Philadelphia, Pennsylvania.

[7] J. Sun., 1993, A convergence proof for an affine-scaling method for convex quadratic programming without nondegeneracy assumptions, Math. Program. 60, 69-79.

[8] P.Tseng., 2004, Convergence properties of Dikin's affine scaling algorithm for non-convex quadratic minimization, J.Global Optim. 30, 285-300.

[9] Y. Ye.,1992, On an affine scaling algorithm for non-convex quadratic programming, Math. Program. 52, 285-300.