NORTH-HOLLAND

A Stronger Extension of the Hardy Inequality*

Lie-heng Wang and Ya-xiang Yuan
State Key Laboratory of Scientific and Engineering Computing
Institute of Computational Mathematics
Academia Sinica
P.O. Box 2719
Beijing 100080, China

Submitted by Hans Schneider

ABSTRACT

We formulate the Hardy inequality in a special matrix form. Thus we easily obtain a new proof for it. By introducing a slightly perturbed matrix, we establish a stronger inequality. © 1998 Elsevier Science Inc.

1. A NEW PROOF FOR THE HARDY INEQUALITY

In the twenties of this century, the British mathematician G. H. Hardy obtained an inequality as follows [1]:

THEOREM 1. *For any integer $n \geqslant 1$ and reals a_i $(i = 1, 2, \dots, n)$, the following inequality holds*:

$$\sum_{k=1}^{n} \left(\frac{\sum_{i=1}^{k} a_i}{k} \right)^2 < 4 \sum_{k=1}^{n} a_k^2, \tag{1}$$

unless all the a_i are zero. The constant 4 *is the best possible.*

*This project was supported by National Foundation of Natural Science of China.

LINEAR ALGEBRA AND ITS APPLICATIONS 270:275–286 (1998)

0024-3795/98/$19.00
PII S0024-3795(97)00245-0

There have been numerous proofs for the inequality (1). In this paper, we will present a new proof, by means of estimating eigenvalues of matrices. Using this technique, we will obtain a sharper inequality than (1).

Proof of the inequality (1). The left hand side of (1) can be rewritten as follows:

$$\sum_{k=1}^{n}\left(\frac{\sum_{i=1}^{k} a_i}{k}\right)^2 = \sum_{k=1}^{n}\sum_{i,j=1}^{k}\frac{a_i a_j}{k^2} = \sum_{i,j=1}^{n}\alpha_{ij}a_i a_j, \tag{2}$$

where

$$\alpha_{ij} = \sum_{k=\max(i,j)}^{n}\frac{1}{k^2}. \tag{3}$$

Define the matrix

$$A = [\alpha_{ij}]_{1\leqslant i,j\leqslant n}. \tag{4}$$

Then by direct calculation, it can be seen that A^{-1} is a symmetric tridiagonal matrix with $(A^{-1})_{ii} = i^2 + (i-1)^2$ and $(A^{-1})_{i\,i+1} = -i^2$, namely

$$A^{-1} = \begin{pmatrix} 1 & -1 & 0 & \cdots & \cdots & 0 & 0 \\ -1 & 1^2+2^2 & -2^2 & \cdots & \cdots & 0 & 0 \\ 0 & 0 & \ddots & \ddots & & \vdots & \vdots \\ \vdots & \vdots & & \ddots & \ddots & 0 & 0 \\ 0 & 0 & \cdots & \cdots & -(n-2)^2 & (n-2)^2+(n-2)^2 & -(n-1)^2 \\ 0 & 0 & \cdots & \cdots & 0 & -(n-1)^2 & (n-1)^2+n^2 \end{pmatrix}. \tag{5}$$

It is obvious that A is a positive definite matrix. Let Δ_k denote the kth principal minor determinant of the matrix $A^{-1} - \lambda I$; then

$$\Delta_{k+1} = \left[k^2 + (k+1)^2 - \lambda\right]\Delta_k - k^4\Delta_{k-1}, \qquad 1 \leqslant k \leqslant n-1, \tag{6}$$

with $\Delta_0 = 1$. Define

$$x_k = \Delta_k/\Delta_{k-1}, \qquad 1 \leqslant k \leqslant n. \tag{7}$$

The relation (6) gives that

$$x_{k+1} = \left[k^2 + (k+1)^2 - \lambda\right] - \frac{k^4}{x_k}, \qquad 1 \leqslant k \leqslant n-1, \tag{8}$$

which can be further rewritten as

$$P_{k+1} = 1 + \left(\frac{k}{k+1}\right)^2 \left(1 - \frac{1}{P_k}\right) - \frac{\lambda}{(k+1)^2}, \qquad 1 \leqslant k \leqslant n-1 \tag{9}$$

if we use the notation

$$P_k = x_k / k^2. \tag{10}$$

Finally let

$$\theta_k = k(1 - P_k); \tag{11}$$

it follows from (9) that

$$\theta_{k+1} = \frac{\lambda}{k+1} + \frac{k}{k+1} \frac{\theta_k}{1 - (\theta_k / k)}, \qquad 1 \leqslant k \leqslant n-1. \tag{12}$$

Let $\lambda = \frac{1}{4}$, and noting that $\Delta_0 = 1$, we have that

$$\Delta_1 = \tfrac{3}{4}, \qquad x_1 = \tfrac{3}{4}, \qquad P_1 = \tfrac{3}{4}, \qquad \theta_1 = \tfrac{1}{4}.$$

Therefore, in order to prove the inequality (1), it is sufficient to prove that the maximum eigenvalue of A is less than 4, i.e., the minimum eigenvalue of the matrix A^{-1} is greater than $\frac{1}{4}$, which is equivalent to proving that the matrix $A^{-1} - \frac{1}{4}I$ is positive definite. Thus, it is sufficient to prove that all the principal minor determinants of the matrix $A^{-1} - \frac{1}{4}I$ are positive. From (7) and the fact that $\Delta_0 = 1$, it is sufficient to prove that $x_k > 0$ $(1 \leqslant k \leqslant n)$, which is equivalent to $P_k > 0$ $(1 \leqslant k \leqslant n)$, or $\theta_k < k$ $(1 \leqslant k \leqslant n)$ for $\lambda = \frac{1}{4}$. In fact, we can prove a sharper inequality for θ_k:

LEMMA 1. *For $\lambda = \frac{1}{4}$, the following inequalities hold:*

$$\frac{1}{4} \leqslant \theta_k \leqslant \frac{1}{2} - \frac{1}{4k}, \qquad 1 \leqslant k \leqslant n, \tag{13}$$

where θ_k is defined by the iterative relation (12), *with $\theta_1 = \frac{1}{4}$.*

Proof. First we prove the second inequality of (13) by induction. Since $\theta_1 = \frac{1}{4}$, we have $\theta_k \leqslant \frac{1}{2} - 1/4k$ for $k = 1$. Assuming that the second inequality of (13) is true for k, we now prove that this inequality is also true for $k + 1$. In fact, from (12) with $\lambda = \frac{1}{4}$, it can be seen that

$$\begin{aligned}
\theta_{k+1} &= \frac{1}{4(k+1)} + \left(\frac{k}{k+1}\right)\frac{\theta_k}{1 - (\theta_k/k)} = \frac{1}{4(k+1)}\left(1 + \frac{4k^2\theta_k}{k - \theta_k}\right) \\
&\leqslant \frac{1}{4(k+1)}\left(1 + \frac{4k^2(\frac{1}{2} - 1/4k)}{k - (\frac{1}{2} - 1/4k)}\right) \\
&= \frac{1}{2}\left(1 - \frac{1}{2(k+1)}\frac{4k^2+1}{4k^2 - 2k + 1}\right) \\
&\leqslant \frac{1}{2}\left(1 - \frac{1}{2(k+1)}\right) \qquad (k \geqslant 1).
\end{aligned}$$

Next, the first inequality of (13) will be proved also by induction. Since $\theta_1 = \frac{1}{4}$, the inequality is true for $k = 1$. Assuming that it is true for k, we now prove that it is also true for $k + 1$. In fact, from (12) with $\lambda = \frac{1}{4}$ it can be seen that

$$\theta_{k+1} \geqslant \frac{1}{4(k+1)}\left(1 + \frac{4k^2 \cdot \frac{1}{4}}{k - \frac{1}{4}}\right) = \frac{1}{4} \cdot \frac{4k^2 + 4k - 1}{4k^2 + 3k - 1} \geqslant \frac{1}{4} \qquad (k \geqslant 1).$$

The proof is completed. ∎

The second inequality of (13) implies that the inequality (1) holds. Thus we have given a new proof for the Hardy inequality.

2. THE PROOF THAT THE CONSTANT 4 IS THE BEST POSSIBLE

In order to prove that the constant 4 in (1) is the best possible, it is sufficient that for any $\lambda > \frac{1}{4}$, the matrix $A^{-1} - \lambda I$ is not positive definite, i.e., there exists a negative principal minor determinant of the matrix $A^{-1} - \lambda I$, for n large enough. We prove it by contradiction. If the constant 4 is not the smallest constant that makes (1) hold for all n, there exists $\lambda > \frac{1}{4}$ such that $A^{-1} - \lambda I$ is positive definite for all n. Again using the notation of Section 1, we have that $\theta_k < k \ \forall k \geqslant 1$. Rewrite (12) as follows:

$$\theta_{k+1} = \theta_k + \frac{(\theta_k/k)(\theta_k - \lambda) + \left(\theta_k - \frac{1}{2}\right)^2 + \left(\lambda - \frac{1}{4}\right)}{(k+1)(1 - \theta_k/k)}. \tag{14}$$

Noting that $\theta_1 = \lambda < 1$, by induction it can be seen that the sequence $\{\theta_k\}$ is montonically increasing, $\theta_k \geqslant \lambda$, and

$$\theta_{k+1} \geqslant \theta_k + \frac{(\theta_k/k)(\theta_k - \lambda) + \left(\theta_k - \frac{1}{2}\right)^2 + \left(\lambda - \frac{1}{4}\right)}{k+1}. \tag{15}$$

First, from (15) we have

$$\theta_{k+1} \geqslant \theta_k + \frac{\lambda - \frac{1}{4}}{k+1},$$

which, together with $\lambda > \frac{1}{4}$, implies that

$$\theta_k \geqslant \theta_1 + \left(\lambda - \tfrac{1}{4}\right) \sum_{s=1}^{k} \frac{1}{s} \to \infty, \qquad k \to \infty. \tag{16}$$

Next, again from (15), using (16), we have, for k large enough,

$$\theta_{k+1} \geqslant \theta_k + \frac{1}{k+1}\left(\theta_k - \tfrac{1}{2}\right)^2 \geqslant \theta_k + \frac{\theta_k^2}{2k}. \tag{17}$$

The above inequality yields that

$$\frac{\theta_{k+1}}{k+1} \geqslant \frac{\theta_k}{k+1} + \frac{\theta_k^2}{2k(k+1)}$$

$$= \frac{\theta_k}{k} + \frac{\theta_k^2}{2k^2} - \frac{\theta_k}{k(k+1)} - \frac{\theta_k^2}{2k^2(k+1)}. \tag{18}$$

Since $\theta_k \to \infty$ as $k \to \infty$, we have $\theta_k < \frac{1}{6}\theta_k^2$ for sufficiently large k. Thus it follows from (18) that

$$\frac{\theta_{k+1}}{k+1} \geqslant \frac{\theta_k}{k} + \frac{\theta_k^2}{2k^2} - \left(\frac{\theta_k^2}{6k(k+1)} + \frac{\theta_k^2}{2k^2(k+1)}\right)$$

$$= \frac{\theta_k}{k} + \frac{\theta_k^2}{2k^2} - \frac{(k+3)\theta_k^2}{6k^2(k+1)} \geqslant \frac{\theta_k}{k} + \frac{\theta_k^2}{4k^2}$$

for k large enough. Let $q_k = \theta_k/k$; then

$$q_{k+1} \geqslant q_k + \tfrac{1}{4}q_k^2. \tag{19}$$

Therefore, noting that $q_k > 0$, it can be seen that the sequence $\{q_k\}$ is monotonically increasing, and

$$q_k \to \infty \qquad \text{as} \quad k \to \infty,$$

since otherwise $q_k \to q < \infty$ as $k \to \infty$, and then by (19) we would have

$$q \geqslant q + \tfrac{1}{4}q^2,$$

which gives $q = 0$. This is a contradiction because $q_k > 0$ and the sequence $\{q_k\}$ is monotonously increasing. Thus for k sufficiently large,

$$\theta_k > k,$$

which contradicts the assumption that the matrix $A^{-1} - \lambda I$ is positive. Therefore the constant 4 cannot be improved.

3. A STRONGER EXTENSION OF THE HARDY INEQUALITY

In this section we present an improvement of the Hardy inequality (1). To do this, we introduce a symmetric, positive definite matrix B^{-1} as follows:

$$B^{-1} = \begin{pmatrix} 1 & -1 & 0 & \cdot & \cdot & \cdot & 0 \\ -1 & 1^2+2^2 & -2^2 & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & & & \cdot \\ \cdot & & \cdot & \cdot & \cdot & & \cdot \\ \cdot & & & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot & -(n-2)^2 & (n-2)^2+(n-1)^2 & -(n-1)^2 \\ 0 & \cdot & \cdot & \cdot & 0 & -(n-1)^2 & (n-1)^2+\delta \end{pmatrix}. \tag{20}$$

The difference between B^{-1} and A^{-1} is only in the last diagonal elements. We now determine the parameter $\delta > 0$ such that the smallest eigenvalue of B^{-1} is also larger than $\frac{1}{4}$.

Let the ith principal minor determinant of the matrix $B^{-1} - \frac{1}{4}I$ be denoted by $\tilde{\Delta}_i$; then $\tilde{\Delta}_i = \Delta_i$ $(i = 1, \dots, n-1)$, where Δ_i is the ith principal minor determinant of the matrix $A^{-1} - \frac{1}{4}I$. In Section 1, it is proved that $\Delta_i > 0$ $(i = 1, \dots, n-1)$, which shows that $\tilde{\Delta}_i > 0$ $(i = 1, \dots, n-1)$. Thus it is sufficient to prove that $\tilde{\Delta}_n > 0$. It is easily seen that

$$\tilde{\Delta}_n = \left[(n-1)^2 + \delta - \tfrac{1}{4}\right]\tilde{\Delta}_{n-1} - (n-1)^4\tilde{\Delta}_{n-2}. \tag{21}$$

Using the same notation as in Section 1, let

$$\tilde{x}_k = \tilde{\Delta}_k / \tilde{\Delta}_{k-1}. \tag{22}$$

Noting that $\tilde{x}_k = x_k = \Delta_k / \Delta_{k-1}$, $k = 1, 2, \dots, n-1$ (see Section 1), then

$$\tilde{x}_n = \left[(n-1)^2 + \delta - \tfrac{1}{4}\right] - \frac{(n-1)^2}{1 - \theta_{n-1}/n-1}. \tag{23}$$

By Lemma 1,

$$\theta_{n-1} \leqslant \frac{1}{2} - \frac{1}{4(n-1)}.$$

Thus in order to make that $B^{-1} - \frac{1}{4}I$ positive definite, it is sufficient that $\tilde{x}_n > 0$. Consequently it is sufficient that

$$\tilde{x}_n \geqslant \left[(n-1)^2 + \delta - \tfrac{1}{4}\right] - \frac{(n-1)^2}{1 - \dfrac{1}{2(n-1)} + \dfrac{1}{4(n-1)^2}} > 0, \quad (24)$$

which is equivalent to

$$\delta > \tfrac{1}{4} - (n-1)^2 + \frac{(n-1)^2}{1 - \dfrac{1}{n-1}\left(\dfrac{1}{2} - \dfrac{1}{4(n-1)}\right)}$$

$$= \frac{8n^3 - 24n^2 + 22n - 5}{16n^2 - 40n + 28}, \quad (25)$$

or

$$\frac{1}{\delta} < \frac{16n^2 - 40n + 28}{8n^3 - 24n^2 + 22n - 5}. \quad (26)$$

Let $y = 1/n$. For n large enough, we have the following expansion in power series:

$$\frac{16y - 40y^2 + 28y^3}{8 - 24y + 22y^2 - 5y^3} = a_1 y + a_2 y^2 + a_3 y^3 + \cdots, \quad (27)$$

where the coefficients $a_1 = 2$, $a_2 = 1$, and $a_3 = 1$. It can be easily calculated that

$$\frac{16n^2 - 40n + 28}{8n^3 - 24n^2 + 22n - 5} - \left(\frac{2}{n} + \frac{1}{n^2} + \frac{1}{n^3}\right)$$

$$= \frac{12n^2 - 17n + 5}{n^3(8n^3 - 24n^2 + 22n - 5)} \geqslant 0, \quad (28)$$

where the equality holds iff $n = 1$.

Thus we have obtained the following result:

LEMMA 2. *If the parameter $\delta > 0$ is taken such that*

$$\frac{1}{\delta} = \frac{2}{n} + \frac{1}{n^2} + \frac{1}{n^3}, \tag{29}$$

then the smallest eigenvalue of the matrix B^{-1} is larger than or equal to $\frac{1}{4}$, and the equality holds iff $n = 1$.

From the expressions (5) and (24), it can be seen that the elements of the symmetric, positive definite matrices B^{-1} and A^{-1} are the same, except that the last diagonal element of B^{-1} is less than that of A^{-1}. So the eigenvalues of B^{-1} are less than the corresponding ones of A^{-1}. The inequality (1) is equivalent to the statement that the minimum eigenvalue of A^{-1} is larger than $\frac{1}{4}$. Thus it is reasonable to think that the minimum eigenvalue of B^{-1} being larger than $\frac{1}{4}$ should be equivalent to an inequality which is stronger than the inequality (1). That will be proved in the following. To do so, we need to write the explicit expression for B.

From (20), we have

$$B^{-1} = A^{-1} - \left(n^2 - \frac{n}{2}\right)e_n e_n^T, \tag{30}$$

where

$$e_n = (0, 0, \ldots, 0, 1)^T. \tag{31}$$

The following lemma is well known:

LEMMA 3. *Assume that $A \in R^{n \times n}$ is a nonsingular matrix, e is an n dimensional column vector, and $\sigma \in R$ such that*

$$1 - \sigma e^T A^{-1} e \neq 0. \tag{32}$$

Then $B = A - \sigma e e^T$ is nonsingular and B^{-1} can be written as follows:

$$B^{-1} = A^{-1} + \frac{\sigma A^{-1} e e^T A^{-1}}{1 - \sigma e^T A^{-1} e}. \tag{33}$$

By the above lemma, it follows from (30) and the expression (3) for the matrix $A = (\alpha_{ij})$ that

$$B = \left[A^{-1} - (n^2 - \delta)e_n e_n^T\right]^{-1} = A + \frac{(n^2 - \delta)Ae_n e_n^T A}{1 - (n^2 - \delta)e_n^T A e_n}$$

$$= A + \frac{(n^2 - \delta)\begin{pmatrix} \frac{1}{n^2} \\ \cdot \\ \cdot \\ \cdot \\ \frac{1}{n^2} \end{pmatrix}\left(\frac{1}{n^2}, \dots, \frac{1}{n^2}\right)}{1 - (n^2 - \delta)\frac{1}{n^2}}$$

$$= A + \left(\frac{1}{\delta} - \frac{1}{n^2}\right)\begin{pmatrix} 1 \\ \cdot \\ \cdot \\ \cdot \\ 1 \end{pmatrix}(1, \dots, 1). \tag{34}$$

THEOREM 2 (A stronger extension of the Hardy inequality). *For any integer $n \geqslant 1$ and real numbers a_i $(1 \leqslant i \leqslant n)$, the following inequality holds:*

$$\sum_{k=1}^{n}\left(\frac{\sum_{i=1}^{k} a_i}{k}\right)^2 + \left(\frac{2}{n} + \frac{1}{n^3}\right)\left(\sum_{k=1}^{n} a_k\right)^2 \leqslant 4\sum_{k=1}^{n} a_k^2; \tag{35}$$

the equality holds iff $n = 1$, unless all the a_i are zero.

Proof. From Lemma 2, the eigenvalues of the matrix B are less than or equal to 4 for $\delta^{-1} = 2/n + 1/n^2 + 1/n^3$; then

$$a^T B a \leqslant 4\|a\|^2,$$

where $a = (a_1, \dots, a_n)^T$, $\|a\|^2 = a^T a$. By the expression (34), we have

$$a^T A a + \left(\frac{2}{n} + \frac{1}{n^3}\right) a^T \begin{pmatrix} 1 \\ \cdot \\ \cdot \\ \cdot \\ 1 \end{pmatrix} (1, \dots, 1) a \leqslant 4\|a\|^2;$$

the equality holds iff $n = 1$, in which case the term $a^T A a$ is just the left hand side of the Hardy inequality (1). Thus the proof is completed. ∎

REMARK 1. If the parameter δ is taken such that $1/\delta = 2/n + 1/n^2$, then we obtain the following generalized Hardy inequality (cf. [2, p. 131]):

$$\sum_{k=1}^{n} \left(\frac{\sum_{i=1}^{k} a_i}{k} \right)^2 + \frac{2}{n} \left(\sum_{k=1}^{n} a_k \right)^2 < 4 \sum_{k=1}^{n} a_k^2. \tag{36}$$

REMARK 2. By direct calculation, we can obtain the following expansion, which is more precise than the expansion (28):

$$\begin{aligned} &\frac{16n^2 - 40n + 28}{8n^3 - 24n^2 + 22n - 5} \\ &\quad = \frac{2}{n} + \frac{1}{n^2} + \frac{1}{n^3} + \frac{3}{2n^4} + \frac{19}{8n^5} + \frac{29}{8n^6} + \frac{169}{32n^7} + \frac{471}{64n^8} \\ &\quad\quad + \frac{1257}{128n^9} + \frac{1603}{128n^{10}} + \frac{1941}{128n^{11}} + \frac{17603}{1024n^{12}} \\ &\quad\quad + \frac{144976n^2 - 309626n + 88015}{1024n^{12}(8n^3 - 24n^2 + 22n - 5)} \\ &\quad = \frac{1}{\delta(n)} + \Delta(n). \end{aligned} \tag{37}$$

It can be easily seen that $\Delta(n) > 0$ for all $n \geqslant 2$, and then, as in the above argument, we can obtain the following improved Hardy inequality: for all $n \geqslant 2$,

$$\sum_{k=1}^{n} \left(\frac{\sum_{i=1}^{k} a_i}{k} \right)^2 + \left(\frac{1}{\delta(n)} - \frac{1}{n^2} \right) \left(\sum_{k=1}^{n} a_k \right)^2 < 4 \sum_{k=1}^{n} a_k^2. \tag{38}$$

REFERENCES

1 G. H. Hardy, J. E. Littlewood, and G. Polya, *Inequalities*, Cambridge U.P., 1964 (reprint of 2nd ed.; 1st ed., Cambridge U.P., 1934).
2 D. S. Mitrinovic, *Analytic Inequalities*, Springer-Verlag, Berlin, 1970.

Received 3 February 1997; final manuscript accepted 24 March 1997